GOVERNMENT OF TAMIL NADU

BUSINESS MATHEMATICS AND STATISTICS

HIGHER SECONDARY FIRST YEAR

VOLUME - II

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Department of School Education

Untouchability is Inhuman and a Crime

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HOW TO USE THE BOOK

Career Options	List of Further Studies & Professions.
Learning Objectives:	Learning objectives are brief statements that describe what students will be expected to learn by the end of school year, course, unit, lesson or class period.
	Additional information about the concept.
	Amazing facts, Rhetorical questions to lead students to Mathematical inquiry
Exercise	Assess students' critical thinking and their understanding
ICT	To enhance digital skills among students
Web links	List of digital resources
	To motivate the students to further explore the content digitally and take them in to virtual world
Miscellaneous Problems	Additional problems for the students
Glossary	Tamil translation of Mathematical terms
References	List of related books for further studies of the topic

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CAREER OPTIONS IN BUSINESS MATHEMATICS AND STATISTICS

Higher Secondary students who have taken commerce with Business Mathematics and Statistics can take up careers in BCA, B.Com., and B.Sc. Statistics. Students who have taken up commerce stream, have a good future in banking and financial institutions.

A lot of students choose to do B.Com with a specialization in computers. Higher Secondary Commerce students planning for further studies can take up careers in professional fields such as Company Secretary , Chartered Accountant (CA), ICAI and so on. Others can take up bachelor's degree in commerce (B.Com), followed by M.Com, Ph.D and M.Phil. There are wide range of career opportunities for B.Com graduates.

After graduation in commerce, one can choose MBA, MA Economics, MA Operational and Research Statistics at Postgraduate level. Apart from these, there are several diploma, certificate and vocational courses which provide entry level jobs in the field of commerce.

Career chart for Higher Secondary students who have taken commerce with Business Mathematics and Statistics.

Courses	Institutions	Scope for further studies
B.Com., B.B.A., B.B.M., B.C.A., B.Com (Computer), B.A.	• Government Arts & Science Colleges, Aided Colleges, Self financing Colleges. • Shri Ram College of Commerce (SRCC), Delhi • Symbiosis Society's College of Arts & Commerce, Pune. • St. Joseph's College, Bangalore	**C.A., I.C.W.A, C.S.**
B.Sc Statistics	• Presidency College, Chepauk, Chennai. • Madras Christian College, Tambaram • Loyola College, Chennai. • D.R.B.C.C Hindu College, Pattabiram, Chennai.	**M.Sc., Statistics**
B.B.A., LLB, B.A., LLB, B.Com., LL.B. (Five years integrated Course)	• Government Law College. • School of excellence, Affiliated to Dr.Ambethkar Law University	**M.L.**
M.A. Economics (Integrated Five Year course) – Admission based on All India Entrance Examination	• Madras School of Economics, Kotturpuram, Chennai.	**Ph.D.,**
B.S.W.	• School of Social studies, Egmore, Chennai	M.S.W

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Assessment

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SYLLABUS

6.	**APPLICATIONS OF DIFFERENTIATION**	**(25 periods)**
	Applications in business and economics: Demand, supply, cost, revenue and profit functions - Elasticity. Maxima and minima: Increasing and Decreasing functions - Stationary Value of a function - Local and Global(Absolute) Maxima and Minima. Applications of Maxima and Minima: Problems on profit maximization and minimization of cost function - Inventory control - Economic Order Quantity (EOQ). Partial Derivatives: Successive partial derivatives - Euler's theorem and its applications. Applications of partial derivatives: Production function and Marginal productivities of two variables - Partial elasticity of demand.	
7.	**FINANCIAL MATHEMATICS**	**(15 periods)**
	Annuities: Types of annuities - Stocks, shares, debentures and Brokerage: Types of shares- Definitions.	
8.	**DESCRIPTIVE STATISTICS AND PROBABILITY**	**(25 periods)**
	Measures of central tendency: Recall – Average - Related Positional Measures - Quartiles, Deciles and Percentiles - Computations for Related positional measure - Geometric mean - Harmonic mean. Measures of dispersion: Quartile Deviation - Mean deviation. Probability: Recall - Basic concepts of Probability - Independent and Dependent events - Conditional Probability - Baye's Theorem.	
9.	**CORRELATION AND REGRESSION ANALYSIS**	**(20 periods)**
	Correlation: Meaning of Correlation - Types of correlation - Scatter Diagram - Karl Pearson's Correlation Coefficient. Rank correlation: Spearman's Rank Correlation Coefficient. Regression Analysis: Dependent and independent variables - Regression Equations.	
10.	**OPERATIONS RESEARCH**	**(20 periods)**
	Linear programming problem: Mathematical formulation of a linear programming problem - Solution of LPP by graphical method. Network Analysis: Construction of network - Critical path analysis.	

Chapter 6

APPLICATIONS OF DIFFERENTIATION

Learning Objectives

After studying this chapter, the students will be able to understand

- the concepts of demand and supply
- meaning and uses of cost, revenue and profit function
- average and marginal concepts
- elasticity of demand and supply
- relationship among average revenue, marginal revenue and elasticity of demand
- application of increasing and decreasing functions
- application of maxima and minima
- concept of Economic Order Quantity
- concepts of partial differentiation
- applications of partial derivatives in Economics
- partial elasticities of demand

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Introduction

Modern economic theory is based on both differential and integral calculus. In economics, differential calculus is used to compute marginal cost, marginal revenue, maxima and minima, elasticities, partial elasticities and also enabling economists to predict maximum profit (or) minimum loss in a specific condition. In this chapter, we will study about some important concepts and applications of differentiation in business and economics.

Leonhard Euler

Euler's theorem in Economics

If factors of production function are paid as factors times of their marginal productivities, then the total factor payment is equal to the degree of homogeneity times the production function.

6.1 Applications of differentiation in business and economics

In an economic situation, consider the variables are price and quantity. Let p be the unit price in rupees and x be the production (output / quantity) of a commodity demanded by the consumer (or) supplied by the producer.

6.1.1 Demand, supply, cost, revenue and profit functions

Demand function

In a market, the quantity of a commodity demanded by the consumer depends on its price. If the price of the commodity increases, then the demand decreases and if the price of the commodity decreases, then the demand increases.

The relationship between the quantity and the unit price of a commodity demanded by consumer is called as **demand function** and is defined as $x = f(p)$ or $p = f(x)$, where $x>0$ and $p>0$.

Graph of the demand function, $\boldsymbol{x = f(p)}$

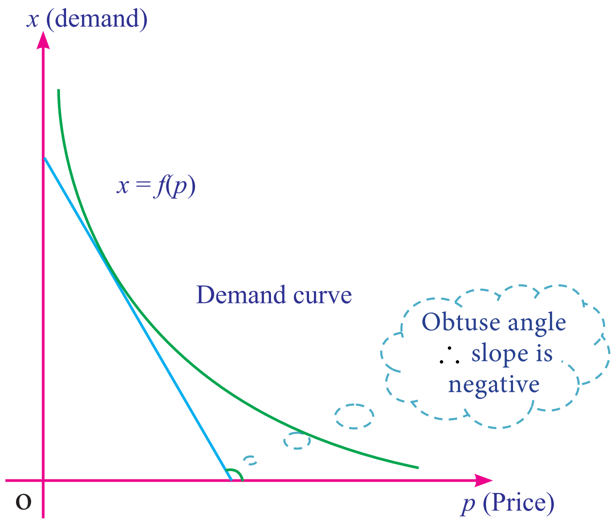


Fig : 6.1

DO YOU KNOW?

The "demand - price relationship" curve illustrates the negative relationship between price and quantity demanded.

Observations

(i) Price and quantity of the demand function are in inverse variation.

(ii) The graph of the demand function lies only in first quadrant.

(iii) Angle made by any tangent to the demand curve with respect to the positive direction of x – axis is always an obtuse angle.

(iv) Slope of the demand curve is negative(–ve).

Supply function

In a market, the quantity of a commodity supplied by producer depends on its price. If the price of the commodity increases, then quantity of supply increases and if the price of the commodity decreases, then quantity of supply decreases.

The relationship between the quantity and the unit price of a commodity supplied by producer is called as **supply function** and is defined as $x = g(p)$ or $p=g(x)$ where $x > 0$ and $p > 0$

The graph of the supply function, $x= g(p)$

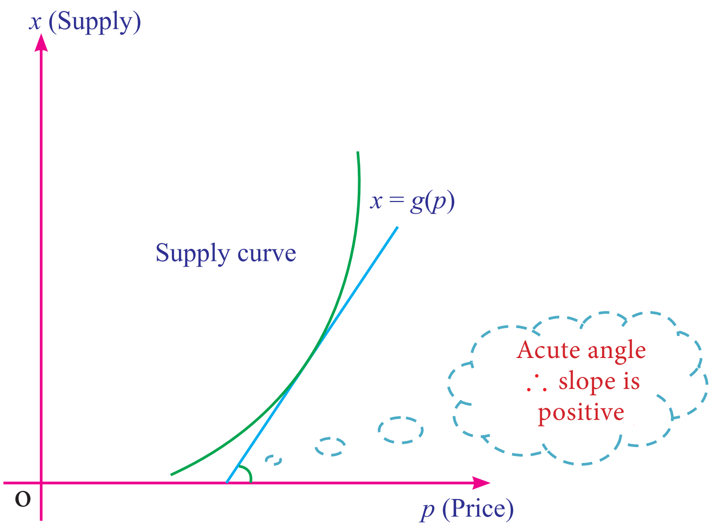


Fig : 6.2

DO YOU KNOW?

The "supply - price relationship" curve illustrates the positive relationship between price and quantity supplied.

Observations

(i) Price and quantity of the supply function are in direct variation.

(ii) The graph of supply function lies only in first quadrant.

(iii) Angle made by any tangent to the supply curve with respect to positive direction of x – axis is always an acute angle.

(iv) Slope of the supply curve is positive (+ve).

DO YOU KNOW?

The law of demand / supply tells us the direction of change, but not the rate at which the change takes place

Equilibrium Price

The price at which the demand for a commodity is equal to its supply is called as Equilibrium Price and is denoted by p_E.

Equilibrium Quantity

The quantity at which the demand for a commodity is equal to its supply is called as Equilibrium Quantity and is denoted by x_E.

NOTE

Usually the demand and supply functions are expressed as x in terms of p, so the equilibrium quantity is obtained either from the demand function (or) from the supply function by substituting the equilibrium price.

Equilibrium Point

The point of intersection of the demand and supply function (p_E, x_E) is called as equilibrium point.

Diagrammatical explanation of equilibrium price, equilibrium quantity and equilibrium point

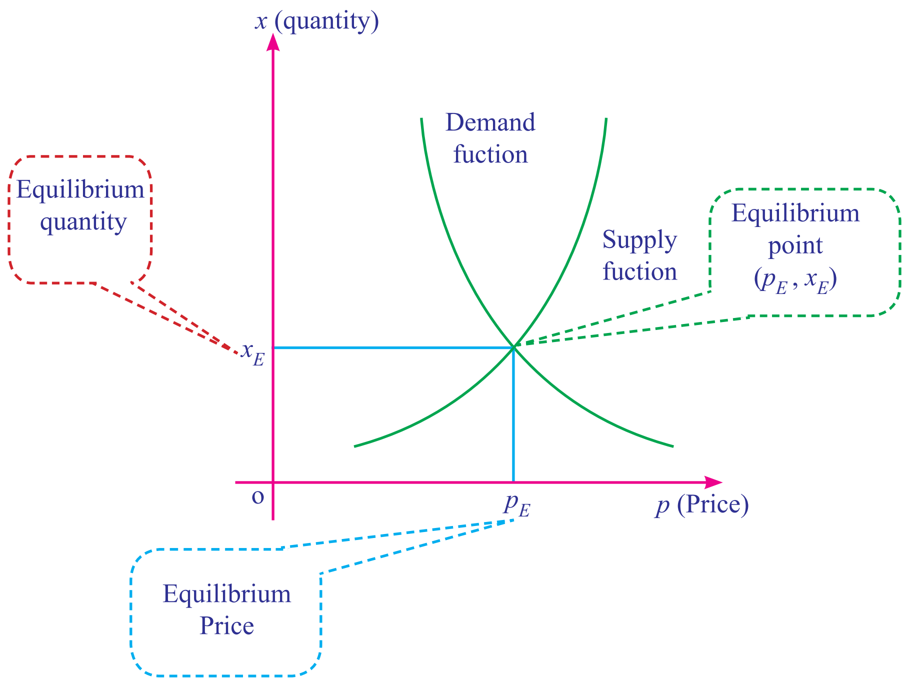


Fig: 6.3

Average and Marginal concepts

Usually, the variation in the dependent quantity 'y' with respect to the independent quantity 'x' can be described in terms of two concepts namely

(i) Average concept and (ii) Marginal concept.

(i) Average concept

The average concept expressed as the variation of y over a whole range of x and is denoted by $\frac{y}{x}$.

(ii) Marginal concept

The marginal concept expressed as the instantaneous rate of change of y with respect to x and is denoted by $\frac{dy}{dx}$.

Remark:

If Δx be the small change in x and Δy be the corresponding change in y of the function $y=f(x)$, then $\frac{\Delta y}{\Delta x}=\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$

Instantaneous rate of change of y with respect to x is defined as the limiting case of ratio of the change in y to the change in x.

i.e. $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$

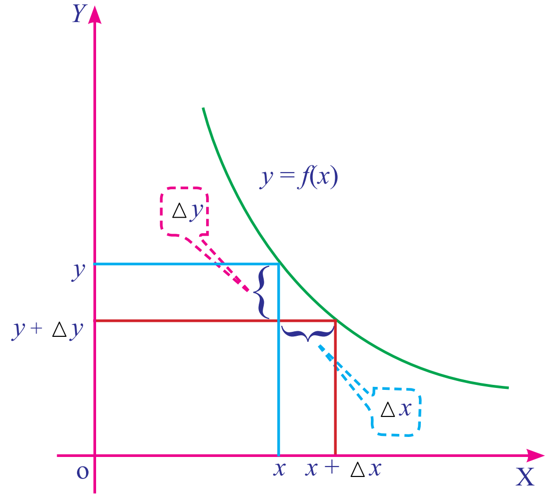


Fig: 6.4(a)

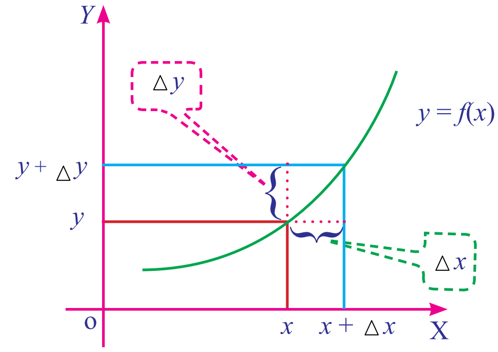


Fig : 6.4(b)

Cost function

The amount spent for the production of a commodity is called its cost function. Normally, total cost function [TC] consists of two parts.

(i) Variable cost (ii) Fixed cost

Variable cost

Variable cost is the cost which varies almost in direct proportion to the volume of production.

Fixed cost

Fixed cost is the cost which does not vary directly with the volume of production.

If $f(x)$ be the variable cost and k be the fixed cost for production of x units, then total cost is $C(x) = f(x) + k$, $x>0$.

NOTE

(i) Variable cost $f(x)$ is a single valued function.

(ii) Fixed cost k is independent of the level of output.

(iii) $f(x)$ does not contain constant term.

Some standard results

If $C(x) = f(x) + k$ be the total cost function, then

(i) Average cost [AC] $= \dfrac{Total\ Cost}{Output} = \dfrac{C(x)}{x} = \dfrac{f(x)+k}{x}$

(ii) Average variable cost [AVC]$= \dfrac{Variable\ Cost}{Output} = \dfrac{f(x)}{x}$

(iii) Average fixed cost [AFC]$= \dfrac{Fixed\ Cost}{Output} = \dfrac{k}{x} =$

(iv) Marginal cost [MC]$= \dfrac{dC}{dx} = \dfrac{d}{dx}[C(x)]= C'(x)$

(v) Marginal average cost [MAC]$= \dfrac{d}{dx}(AC)$

(vi) Total cost [TC]=Average cost $\times$ output

(vi) Average cost [AC] is minimum, when $MC = AC$

Remark:

The marginal cost $[MC]$ is approximately equal to the additional production cost of $(x+1)^{\text{th}}$unit, when the production level is x units.

Diagrammatical explanation of marginal cost $[MC]$

Marginal cost is the change in aggregate cost when the volume of production is increased or decreased by one unit.

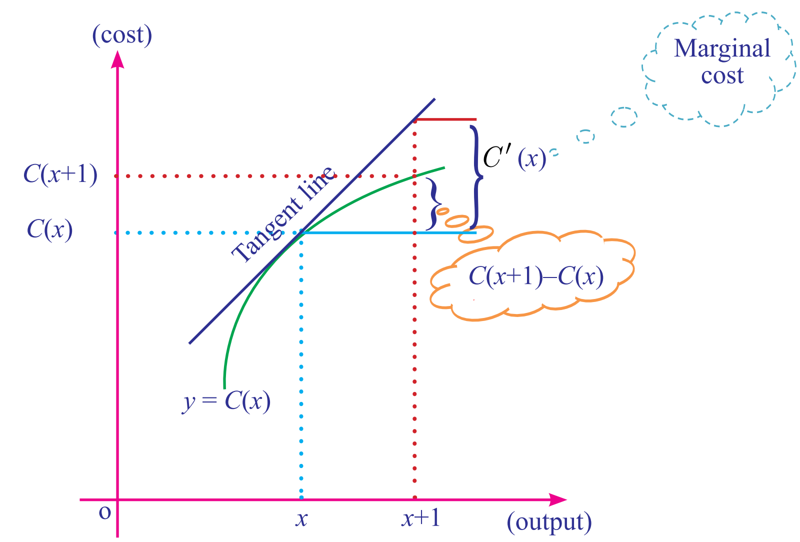


Fig: 6.5

Revenue function

Revenue is the amount realised on a commodity when it is produced and sold. If x is the number of units produced and sold and p is its unit price, then the total revenue function $R(x)$ is defined as $R(x) = px$, where x and p are positive.

DO YOU KNOW?

The revenue increases when the Producer, supplies more quantity at a higher price.

Some standard results

If $R(x) = px$ be the revenue function, then

(i) Average revenue $[AR] = \dfrac{Total\ Revenue}{Output} = \dfrac{R(x)}{x} = p$

(ii) Marginal revenue $[MR] = \dfrac{dR}{dx} = \dfrac{d}{dx}(R(x)) = R'(x)$

(iii) Marginal average revenue $[MAR] = \dfrac{d}{dx}(AR) = AR'(x)$

Remarks:

(i) Average revenue [AR] and price [p] are the same. [i.e. $AR=p$]

(ii) The marginal revenue [MR] is approximately equal to the additional revenue made on selling of $(x+1)^{th}$ unit, when the sales level is x units.

Diagrammatical explanation of Marginal Revenue [*MR*]

Marginal revenue is the change in aggregate revenue when the volume of selling unit is increased by one unit.

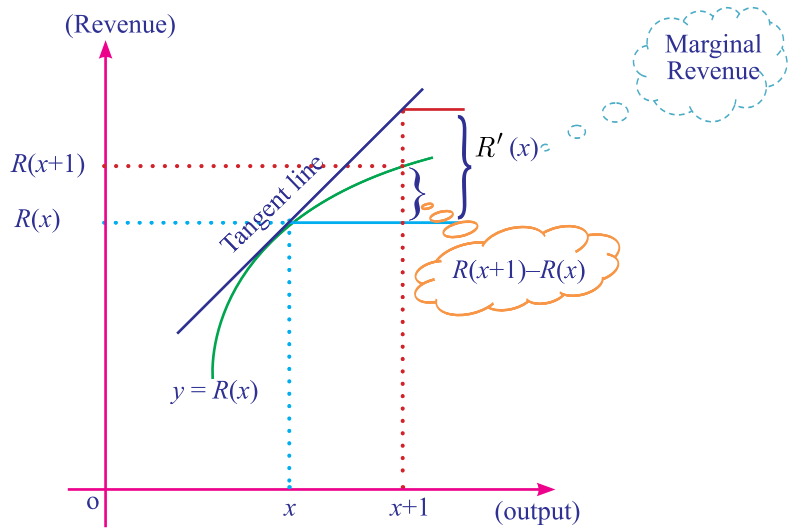


Fig : 6.6

Profit function

The excess of total revenue over the total cost of production is called the profit. If $R(x)$ is the total revenue and $C(x)$ is the total cost, then profit function $P(x)$ is defined as $P(x) = R(x) - C(x)$

Some standard results

If $P(x) = R(x) - C(x)$ be the profit function, then

(i) Average profit $[AP] = \dfrac{Total\ Profit}{Output} = \dfrac{P(x)}{x}$

(ii) Marginal profit $[MP] = \dfrac{dP}{dx} = \dfrac{d}{dx}(P(x)) = P'(x)$

(iii) Marginal average profit $[MAP] = \dfrac{d}{dx}(AP) = AP'(x)$

(iv) Profit $[P(x)]$ is maximum when $MR = MC$

6.1.2 Elasticity

Elasticity 'η' of the function $y = f(x)$ at a point x is defined as the limiting case of ratio of the relative change in y to the relative change in x.

$$\therefore \quad \eta = \frac{E_y}{E_x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\frac{dy}{y}}{\frac{dx}{x}}$$

$$\Rightarrow \quad \eta = \frac{x}{y} \cdot \frac{dy}{dx}$$

DO YOU KNOW?

$$\eta = \frac{\frac{dy}{dx}}{\frac{y}{x}} = \frac{\text{Marginal quantity of } y \text{ with respect to } x}{\text{Average quantity of } y \text{ with respect to } x}$$

(i) Price elasticity of demand

Price elasticity of demand is the degree of responsiveness of quantity demanded to a change in price.

If x is demand and p is unit price of the demand function $x = f(p)$, then the elasticity of demand with respect to the price is defined as $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$.

(ii) Price elasticity of supply :

Price elasticity of supply is the degree of responsiveness of quantity supplied to a change in price.

If x is supply and p is unit price of the supply function $x = g(p)$, then the elasticity of supply with respect to the price is defined as $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$.

NOTE

"price elasticity" is shortly called as "elasticity".

DO YOU KNOW?

Elasticity is predominantly used to assess the change in consumer demand as a result of a change in price

Some important results on price elasticity

(i) If | η |>1, then the quantity demand or supply is said to be elastic.

(ii) If | η |=1, then the quantity demand or supply is said to be unit elastic.

(iii) If | η |<1, then the quantity demand or supply is said to be inelastic.

Remarks:

(i) Elastic : A quantity demand or supply is elastic when its quantity responds greatly to changes in its price. Example: Consumption of onion and its price.

(ii) Inelastic : A quantity demand or supply is inelastic when its quantity responds very little to changes in its price. Example: Consumption of rice and its price.

(iii) Unit elastic : A quantity demand or supply is unit elastic when its quantity responds as the same ratio as changes in its price.

Relationship among Marginal revenue [MR], Average revenue [AR] and Elasticity of demand[η_d].

We know that $R(x) = px$

i.e., $R = px$

and $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

Now, $MR = \frac{d}{dx}(R)$

$$= \frac{d}{dx}(px) = p + x\frac{dp}{dx}$$

$$= p\left[1 + \frac{x}{p}\cdot\frac{dp}{dx}\right]$$

$$= p\left[1 + \frac{1}{\frac{p}{x}\cdot\frac{dx}{dp}}\right]$$

$$= p\left[1 - \frac{1}{-\frac{p}{x}\cdot\frac{dx}{dp}}\right]$$

$$= p\left[1 - \frac{1}{\eta_d}\right]$$

i.e. $MR = AR\left[1 - \frac{1}{\eta_d}\right]$ (or)

$$\eta_d = \frac{AR}{AR - MR}$$

Example 6.1

The total cost function for the production of x units of an item is given by

$C(x) = \frac{1}{3}x^3 + 4x^2 - 25x + 7$. Find

(i) Average cost

(ii) Average variable cost

(iii) Average fixed cost

(iv) Marginal cost and

(v) Marginal Average cost

Solution:

$$C(x) = \frac{1}{3}x^3 + 4x^2 - 25x + 7$$

(i) Average cost (AC) $= \frac{C}{x}$

$$= \frac{1}{3}x^2 + 4x - 25 + \frac{7}{x}$$

(ii) Average variable cost (AVC) $= \frac{f(x)}{x}$

$$= \frac{1}{3}x^2 + 4x - 25$$

(iii) Average fixed cost (AFC) $= \frac{k}{x}$

$$= \frac{7}{x}$$

(iv) Marginal cost (MC) $= \frac{dC}{dx}$ (or) $\frac{d}{dx}(C(x))$

$$= \frac{d}{dx}\left[\frac{1}{3}x^3 + 4x^2 - 25x + 7\right]$$

$$= x^2 + 8x - 25$$

(v) Marginal Average cost (MAC) $= \frac{d}{dx}[AC]$

$$= \frac{d}{dx}\left[\frac{1}{3}x^2 + 4x - 25 + \frac{7}{x}\right]$$

$$= \frac{2}{3}x + 4 - \frac{7}{x^2}$$

Example 6.2

The total cost C in Rupees of making x units of product is $C(x) = 50 + 4x + 3\sqrt{x}$. Find the marginal cost of the product at 9 units of output.

Solution:

$$C(x) = 50 + 4x + 3\sqrt{x}$$

$$\text{Marginal cost } (MC) = \frac{dC}{dx} = \frac{d}{dx}\left[\mathrm{C}(x)\right]$$

$$= \frac{d}{dx}\left[50 + 4x + 3\sqrt{x}\right] = 4 + \frac{3}{2\sqrt{x}}$$

When $x = 9$, $\dfrac{dC}{dx} = 4 + \dfrac{3}{2\sqrt{9}}$

$$= 4\frac{1}{2} \text{ (or) ₹}4.50$$

$\therefore$ MC is ₹ 4.50 , when the level of output is 9 units.

Example 6.3

Find the equilibrium price and equilibrium quantity for the following demand and supply functions.

Demand : $x = \frac{1}{2}(5 - p)$ and Supply : $x = 2p - 3$

Solution:

At equilibrium, demand = supply

$$\Rightarrow \quad \frac{1}{2}(5 - p) = 2p - 3$$

$$5 - p = 4p - 6$$

$$\Rightarrow \quad p = \frac{11}{5}$$

$\therefore$ Equilibrium price: $p_E = \dfrac{11}{5}$

Now, put $p = \dfrac{11}{5}$ in $x = 2p - 3$

We get, $x = 2\left(\dfrac{11}{5}\right) - 3 = \dfrac{7}{5}$

$\therefore$ Equilibrium quantity: $x_E = \dfrac{7}{5}$

Example 6.4

For the demand function $x=\frac{20}{p+1}$, find the elasticity of demand with respect to price at a point $p = 3$. Examine whether the demand is elastic at $p = 3$.

Solution:

$$x=\frac{20}{p+1}$$

$$\frac{dx}{dp}=\frac{-20}{(p+1)^2}$$

Elasticity of demand: $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

$$=-\frac{p}{\left(\frac{20}{(p+1)}\right)}\cdot\frac{-20}{(p+1)^2}$$

$$=\frac{p}{p+1}$$

When $p=3$, $\eta_d=\frac{3}{4}$ (or) 0.75

Here $|\eta_d|<1$

$\therefore$ demand is inelastic.

Example 6.5

Find the elasticity of supply for the supply function $x=2p^2-5p+1$

Solution:

$$x=2p^2-5p+1$$

$$\frac{dx}{dp}=4p-5$$

Elasticity of supply: $\eta_s=\frac{p}{x}\cdot\frac{dx}{dp}$

$$=\frac{p}{2p^2-5p+1}\cdot(4p-5)$$

$$=\frac{4p^2-5p}{2p^2-5p+1}$$

Example 6.6

If $y = \frac{2x+1}{3x+2}$ then, obtain the value of elasticity at $x = 1$.

Solution:

$$y = \frac{2x+1}{3x+2}$$

$$\frac{dy}{dx} = \frac{(3x+2)(2)-(2x+1)(3)}{(3x+2)^2}$$

$$= \frac{1}{(3x+2)^2}$$

$$\text{Elasticity: } \eta = \frac{x}{y} \cdot \frac{dy}{dx}$$

$$= \frac{x}{\left(\frac{2x+1}{3x+2}\right)} \cdot \frac{1}{(3x+2)^2}$$

$$= \frac{x}{(2x+1)(3x+2)}$$

When $x = 1$, $\eta = \frac{1}{15}$

Example 6.7

A demand function is given by $xp^n = k$ where n and k are constants. Prove that elasticity of demand is always constant.

Solution:

$$xp^n = k$$

$$x = kp^{-n}$$

$$\frac{dx}{dp} = -nkp^{-n-1}$$

$$\text{Elasticity of demand : } \eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= -\frac{p}{kp^{-n}}\left(-nkp^{-n-1}\right)$$

$= n$, which is a constant.

Example 6.8

For the given demand function $p = 40 - x$, find the value of the output when $\eta_d = 1$

Solution:

$$p = 40 - x$$

$$\therefore \quad x = 40 - p$$

$$\frac{dx}{dp} = -1$$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$$= \frac{40 - x}{x}$$

Given that $\eta_d = 1$

$$\therefore \quad \frac{40 - x}{x} = 1$$

$$2x = 40$$

$$\therefore \quad x = 20 \text{ units.}$$

Example 6.9

Find the elasticity of demand in terms of x for the demand law $p = (a - bx)^{\frac{1}{2}}$. Also find the values of x when elasticity of demand is unity.

Solution:

$$p = (a - bx)^{\frac{1}{2}}.$$

Differentiating with respect to the price 'p', we get

$$1 = \frac{1}{2}(a - bx)^{\frac{-1}{2}}(-b) \cdot \frac{dx}{dp}$$

$$\therefore \quad \frac{dx}{dp} = \frac{2(a - bx)^{\frac{1}{2}}}{-b}$$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$$= -\frac{(a - bx)^{\frac{1}{2}}}{x} \cdot \frac{2(a - bx)^{\frac{1}{2}}}{-b}$$

$$= \frac{2(a - bx)}{bx}$$

When $\eta_d = 1$, $\frac{2(a - bx)}{bx} = 1$

$$2(a - bx) = bx$$

$$\therefore \quad x = \frac{2a}{3b} \text{ units.}$$

Example 6.10

Verify the relationship of elasticity of demand, average revenue and marginal revenue for the demand law $p = 50 - 3x$.

Solution:

$$p = 50 - 3x$$

$$\frac{dp}{dx} = -3$$

$$\therefore \quad \frac{dx}{dp} = -\frac{1}{3}$$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$$= -\frac{50-3x}{x}\left(-\frac{1}{3}\right)$$

$$= \frac{50-3x}{3x} \qquad \ldots (1)$$

Now, Revenue: $R = px$

$$= (50-3x)x$$

$$= 50x - 3x^2$$

Average revenue: $AR = p$

$$= 50 - 3x$$

Marginal revenue: $MR = \frac{dR}{dx}$

$$= 50 - 6x$$

$$\therefore \quad \frac{AR}{AR - MR} = \frac{50-3x}{(50-3x)-(50-6x)}$$

$$= \frac{50-3x}{3x} \qquad \ldots (2)$$

From (1) and (2), we get

$$\eta_d = \frac{AR}{AR - MR}$$

Hence verified.

Example 6.11

Find the elasticity of supply for the supply law $x = \frac{p}{p+5}$ when p = 20 and interpret your result.

Solution:

$$x = \frac{p}{p+5}$$

$$\frac{dx}{dp} = \frac{(p+5)-p}{(p+5)^2}$$

$$= \frac{5}{(p+5)^2}$$

Elasticity of supply: $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$

$$= (p+5)\frac{5}{(p+5)^2}$$

When $p = 20$, $\eta_s = \frac{5}{20+5}$

$$= 0.2$$

Interpretation:

- If the price increases by 1% from p = ₹ 20, then the quantity of supply increases by 0.2% approximately.
- If the price decreases by 1% from p = ₹ 20, then the quantity of supply decreases by 0.2% approximately.

Example 6.12

For the cost function $C = 2x\left(\frac{x+5}{x+2}\right)+7$, prove that marginal cost (MC) falls continuously as the output x increases.

Solution:

$$C = 2x\left(\frac{x+5}{x+2}\right)+7$$

$$= \frac{2x^2+10x}{x+2}+7$$

Marginal cost:

$$MC = \frac{dC}{dx}$$

$$= \frac{d}{dx}\left[\frac{2x^2+10x}{x+2}+7\right]$$

$$= \frac{(x+2)(4x+10)-(2x^2+10x)}{(x+2)^2}$$

$$= \frac{2(x^2+4x+10)}{(x+2)^2}$$

$$= \frac{2\left[(x+2)^2+6\right]}{(x+2)^2}$$

$$= 2\left[1+\frac{6}{(x+2)^2}\right]$$

$\therefore$ when x increases, MC decreases.

Hence proved.

Example 6.13

$\overline{C} = 0.05x^2+16+\frac{100}{x}$ is the manufacturer's average cost function. What is the marginal cost when 50 units are produced and interpret your result.

Solution:

Total cost: $C = AC \times x$

$= \overline{C} \times x$

$= 0.05x^3+16x+100$

Marginal cost: $MC = \frac{dC}{dx}$

$= 0.15x^2+16$

$$\left(\frac{dC}{dx}\right)_{x=50} = 0.15(50)^2+16$$

$= 375+16$

$=$ ₹ 391

Interpretation:

If the production level is increased by one unit from $x = 50$, then the cost of additional unit is approximately equal to ₹ 391.

Example 6.14

For the function $y = x^3 + 19$, find the values of x when its marginal value is equal to 27.

Solution:

$$y = x^3 + 19$$

$$\frac{dy}{dx} = 3x^2 \qquad \dots (1)$$

$$\frac{dy}{dx} = 27 \qquad \dots (2) \text{ [Given]}$$

From (1) and (2), we get

$$3x^2 = 27$$

$$\therefore \quad x = \pm 3$$

Example 6.15

The demand function for a commodity is $p = \frac{4}{x}$,where p is unit price. Find the instantaneous rate of change of demand with respect to price at p=4. Also interpret your result.

Solution:

$$p = \frac{4}{x}$$

$$\Rightarrow \quad x = \frac{4}{p}$$

$$\therefore \quad \frac{dx}{dp} = -\frac{4}{p^2}$$

At $p = 4$, $\frac{dx}{dp} = -\frac{1}{4} = -0.25$

$\therefore$ Rate of change of demand with respect to the price at p = ₹ 4 is -0.25

Interpretation:

When the price increases by 1% from the level of p = ₹ 4, the demand decreases (falls) by 0.25%

Example 6.16

The demand and the cost function of a firm are p = 497-0.2x and C = 25x+10000 respectively. Find the output level and price at which the profit is maximum.

Solution:

We know that profit is maximum when marginal revenue [MR] = marginal cost [MC].

Revenue: $R = px$

$= (497-0.2x)x$

$= 497x-0.2x^2$

$MR = \dfrac{dR}{dx}$

$\therefore \quad MR = 497-0.4x$

Cost: $C = 25x+10000$

$\therefore \quad MC = 25$

$MR = MC \Rightarrow 497-0.4x = 25$

$\Rightarrow \quad 472-0.4x = 0$

$\Rightarrow \quad x = 1180$ units.

Now, $p = 497-0.2x$

at $x = 1180$, $p = 497-0.2(1180)$

$=$ ₹ 261.

Example 6.17

The cost function of a firm is $C = \frac{1}{3}x^3 - 3x^2 + 9x$. Find the level of output ($x>0$) when average cost is minimum.

Solution:

We know that average cost [AC] is minimum when average cost [AC] = marginal cost [MC].

Cost: $C = \frac{1}{3}x^3 - 3x^2 + 9x$

$\therefore \quad AC = \frac{1}{3}x^2 - 3x + 9$ and

$$MC = x^2 - 6x + 9$$

Now, $AC = MC \Rightarrow \frac{1}{3}x^2 - 3x + 9 = x^2 - 6x + 9$

$\Rightarrow$ $2x^2 - 9x = 0$

$\Rightarrow$ $x = \frac{9}{2}$ units. $[\because x > 0]$

Exercise 6.1

1. A firm produces x tonnes of output at a total cost of $C(x) = \frac{1}{10}x^3 - 4x^2 - 20x + 7$ find the (i) average cost (ii) average variable cost (iii) average fixed cost (iv) marginal cost and (v) marginal average cost.

2. The total cost of x units of output of a firm is given by $C = \frac{2}{3}x + \frac{35}{2}$. Find the
   (i) cost when output is 4 units
   (ii) average cost when output is 10 units
   (iii) marginal cost when output is 3 units

3. Revenue function 'R' and cost function 'C' are $R = 14x - x^2$ and $C = x(x^2 - 2)$. Find the (i) average cost, (ii) marginal cost, (iii) average revenue and (iv) marginal revenue.

4. If the demand law is given by $p = 10e^{-\frac{x}{2}}$ then find the elasticity of demand.

5. Find the elasticity of demand in terms of x for the following demand laws and also find the value of x where elasticity is equals to unity.
   (i) $p = (a - bx)^2$ (ii) $p = a - bx^2$

6. Find the elasticity of supply for the supply function $x = 2p^2 + 5$ when p=3

7. The demand curve of a commodity is given by $p = \frac{50 - x}{5}$, find the marginal revenue for any output x and also find marginal revenue at $x = 0$ and $x = 25$?

8. The supply function of certain goods is given by $x = a\sqrt{p - b}$ where p is unit price, a and b are constants with $p > b$ Find elasticity of supply at $p = 2b$.

9. Show that $MR = p\left[1 - \frac{1}{\eta_d}\right]$ for the demand function $p = 400 - 2x - 3x^2$ where p is unit price and x is quantity demand.

10. For the demand function $p = 550 - 3x - 6x^2$ where x is quantity demand and p is unit price. Show that $MR = p\left[1 - \frac{1}{\eta_d}\right]$

11. For the demand function $x = \frac{25}{p^4}$, $1 \le p \le 5$, determine the elasticity of demand.

12. The demand function of a commodity is $p = 200 - \frac{x}{100}$ and its cost is $C=40x+12000$ where p is a unit price in rupees and x is the number of units produced and sold. Determine (i) profit function (ii) average profit at an output of 10 units (iii) marginal profit at an output of 10 units and (iv) marginal average profit at an output of 10 units.

13. Find the values of x, when the marginal function of $y = x^3 + 10x^2 - 48x + 8$ is twice the x.

14. The total cost function y for x units is given by $y = 3x\left(\frac{x+7}{x+5}\right) + 5$. Show that the marginal cost decreases continuously as the output increases.

15. Find the price elasticity of demand for the demand function $x = 10 - p$ where x is the demand and p is the price. Examine whether the demand is elastic, inelastic or unit elastic at $p = 6$.

16. Find the equilibrium price and equilibrium quantity for the following functions.

    Demand: $x = 100 - 2p$ and supply: $x = 3p - 50$

17. The demand and cost functions of a firm are $x = 6000-30p$ and $C = 72000+60x$ respectively. Find the level of output and price at which the profit is maximum.

18. The cost function of a firm is $C = x^3 - 12x^2 + 48x$. Find the level of output (x>0) at which average cost is minimum.

6.2 Maxima and minima

We are using maxima and minima in our daily life as well as in every field such as chemistry, physics, engineering and in economics etc.,

In particular, we can use maxima and minima

(i) To maximize the beneficial values like profit, efficiency, output of a company etc.,

(ii) To minimize the negative values like, expenses, efforts etc.,

(iii) Used in the study of inventory control, economic order quantity etc.,

6.2.1 Increasing and decreasing functions

Before learning the concept of maxima and minima, we will study the nature of the curve of a given function using derivative.

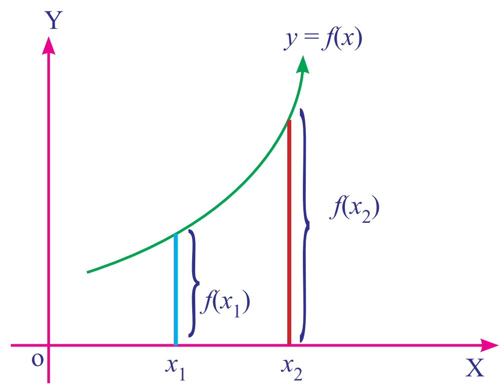
(i) Increasing function

A function $f(x)$ is said to be increasing function in the interval $[a,b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) \le f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$

A function $f(x)$ is said to be strictly increasing in $[a,b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) < f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$



$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Strictly increasing function

Fig: 6.7

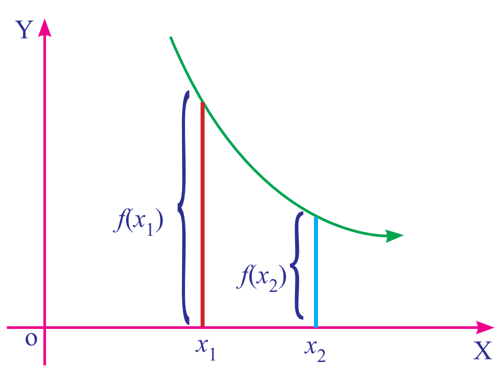
(ii) Decreasing function

A function $f(x)$ is said to be decreasing function in $[a, b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) \ge f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$

A function $f(x)$ is said to be strictly decreasing function in $[a,b]$ if

$$x_1 < x_2 \Rightarrow f\left(x_1\right) > f\left(x_2\right) \text{ for all } x_1, x_2 \in \left[a,b\right]$$



$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Strictly decreasing function

Fig: 6.8

NOTE

A function is said to be monotonic function if it is either an increasing function or a decreasing function.

Derivative test for increasing and decreasing function

Theorem:6.1 (Without Proof)

Let $f(x)$ be a continuous function on $[a,b]$ and differentiable on the open interval (a,b), then

(i) $f(x)$ is increasing in $[a, b]$ if $f'(x) \geq 0$

(ii) $f(x)$ is decreasing in $[a, b]$ if $f'(x) \leq 0$

Remarks:

(i) $f(x)$ is strictly increasing in (a,b) if $f'(x) > 0$ for every $x \in (a,b)$

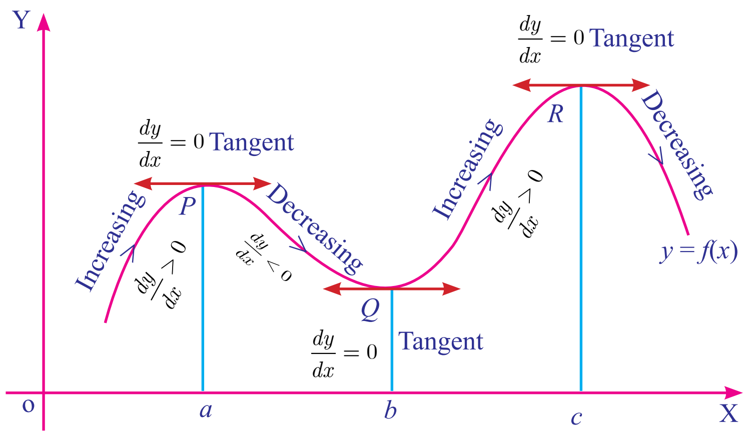
(ii) $f(x)$ is strictly decreasing in (a,b) if $f'(x) < 0$ for every $x \in (a,b)$

(iii) $f(x)$ is said to be a constant function if $f'(x) = 0$

6.2.2 Stationary Value of a function

Let $f(x)$ be a continuous function on $[a, b]$ and differentiable in (a, b). $f(x)$ is said to be stationary at $x = a$ if $f'(a)=0$.

The stationary value of $f(x)$ is $f(a)$. The point $(a, f(a))$ is called stationary point.



P, Q, R are called Stationary Points

Fig: 6.9

In figure 6.9 the function $y = f(x)$ has stationary at $x = a, x = b$ and $x = c$.

At these points, $\frac{dy}{dx} = 0$. The tangents at these points are parallel to x – axis.

NOTE

By drawing the graph of any function related to economics data, we can study the trend of the business related to the function and therefore, we can predict or forecast the business trend.

Example 6.18

Show that the function $f(x) = x^3 - 3x^2 + 4x, x \in R$ is strictly increasing function on R.

Solution :

$$f(x) = x^3 - 3x^2 + 4x, x \in R$$

$$f'(x) = 3x^2 - 6x + 4$$

$$= 3x^2 - 6x + 3 + 1$$

$$= 3(x-1)^2 + 1$$

$$> 0, \text{ for all } x \in R$$

Therefore, the function f is strictly increasing on $(-\infty, \infty)$.

Example 6.19

Find the interval in which the function $f(x) = x^2 - 4x + 6$ is strictly increasing and strictly decreasing.

Solution :

Given that $f(x) = x^2 - 4x + 6$

Differentiate with respect to x,

$$f'(x) = 2x - 4$$

When $f'(x) = 0 \Rightarrow 2x - 4 = 0$

$\Rightarrow$ $x = 2.$

Then the real line is divided into two intervals namely $(-\infty, 2)$ and $(2, \infty)$

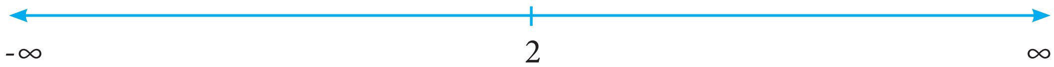


Fig :6.10

[To choose the sign of $f'(x)$ choose any values for x from the intervals ans substitute in $f'(x)$ and get the sign.]

Interval	Sign of $f'(x) = 2x - 4$	Nature of the function
$(-\infty, 2)$	< 0	$f(x)$ is strictly decreasing in $(-\infty, 2)$
$(2, \infty)$	> 0	$f(x)$ is strictly increasing in $(2, \infty)$

Table: 6.1

Example 6.20

Find the intervals in which the function f given by $f(x)=4x^3-6x^2-72x+30$ is increasing or decreasing.

Solution :

$$f(x) = 4x^3-6x^2-72x+30$$

$$f'(x) = 12x^2-12x-72$$

$$= 12(x^2-x-6)$$

$$= 12(x-3)(x+2)$$

$$f'(x) = 0 \Rightarrow 12(x-3)(x+2)=0$$

$$x = 3 \text{ (or) } x = -2$$

$f(x)$ has stationary at $x=3$ and at $x = -2$.

These points divides the whole interval into three intervals namely $(-\infty,-2)$,$(-2,3)$ and $(3,\infty)$.

Fig :6.11

Interval	Sign of $f'(x)$	Intervals of increasing/decreasing
$(-\infty, -2)$	$(-)(-) > 0$	Increasing in $(-\infty, -2]$
$(-2, 3)$	$(-)(+) < 0$	Decreasing in $[-2, 3]$
$(3, \infty)$	$(+)(+) > 0$	Increasing in $[3, \infty)$

Table: 6.2

Example 6.21

Find the stationary value and the stationary points $f(x)=x^2+2x-5$.

Solution :

Given that $\quad f(x) = x^2 + 2x - 5 \quad \ldots (1)$

$f'(x) = 2x + 2$

At stationary points, $\quad f'(x) = 0$

$\Rightarrow \quad 2x + 2 = 0$

$\Rightarrow \quad x = -1$

$f(x)$ has stationary value at $x = -1$

When $\quad x = -1$, from (1)

$f(-1) = (-1)^2+2(-1)-5$

$= -6$

Stationary value of $f(x)$ is -6

Hence stationary point is $(-1,-6)$

Example 6.22

Find the stationary values and stationary points for the function

$f(x)=2x^3+9x^2+12x+1$.

Solution :

Given that $\quad f(x) = 2x^3+9x^2+12x+1.$

$f'(x) = 6x^2+18x+12$

$= 6(x^2+3x+2)$

$= 6(x+2)(x+1)$

$f'(x) = 0 \Rightarrow 6(x+2)(x+1) = 0$

$\Rightarrow \quad x + 2 = 0$ (or) $x + 1 = 0.$

$x = -2$ (or) $x = -1$

$f(x)$ has stationary points at $x = -2$ and $x = -1$

Stationary values are obtained by putting $x = -2$ and $x = -1$

When $\quad x = -2$

$$f(-2) = 2(-8)+9(4)+12(-2)+1$$
$$= -3$$

When $\quad x = -1$

$$f(-1) = 2(-1)+9(1)+12(-1)+1$$
$$= -4$$

The stationary points are $(-2,-3)$ and $(-1,-4)$

Example 6.23

The profit function of a firm in producing x units of a product is given by $P(x)=\frac{x^3}{3}+x^2+x$. Check whether the firm is running a profitable business or not.

Solution :

$$P(x) = \frac{x^3}{3}+x^2+x.$$
$$P'(x) = x^2+2x+1$$
$$=(x+1)^2$$

It is clear that $P'(x)>0$ for all x.

$\therefore$ The firm is running a profitable business.

IMPORTANT NOTE

Let $R(x)$ and $C(x)$ are revenue function and cost function respectively when x units of commodity is produced. If $R(x)$ and $C(x)$ are differentiable for all $x > 0$ then $P(x) = R(x) - C(x)$ is maximized when Marginal Revenue = Marginal cost. That is, when $R'(x)=C'(x)$ profit is maximum at its stationary point.

Example 6.24

Given $C(x)=\frac{x^2}{6}+5x+200$ and $p(x) = 40-x$ are the cost price and selling price when x units of commodity are produced. Find the level of the production that maximize the profit.

Solution :

Given $\quad C(x) = \frac{x^2}{6} + 5x + 200 \quad \ldots (1)$

and $\quad p(x) = 40 - x \quad \ldots (2)$

Profit is maximized when marginal revenue = marginal cost

(i.e) $\quad R'(x) = C'(x)$

$$= \frac{x}{3} + 5$$

$$R = p \cdot x$$

$$= 40x - x^2$$

$$R'(x) = 40 - 2x$$

Hence $\quad 40 - 2x = \frac{x}{3} + 5$

$$x = 15$$

At $x = 15$, the profit is maximum.

6.2.3 Local and Global(Absolute) Maxima and Minima

Definition 6.1

Local Maximum and local Minimum

A function f has a local maximum (or relative maximum) at c if there is an open interval (a,b) containing c such that $f(c) \geq f(x)$ for every $x \in (a,b)$

Similarly, f has a local minimum at c if there is an open interval (a,b) containing c such that $f(c) \leq f(x)$ for every $x \in (a,b)$.

Definition 6.2

Absolute maximum and absolute minimum

A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in domain of f. The number $f(c)$ is called maximum value of f in the domain. Similarly f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in domain of f and the number $f(c)$ is called the minimum value of f on the domain. The maximum and minimum value of f are called extreme values of f.

NOTE

Absolute maximum and absolute minimum values of a function f on an interval (a,b) are also called the global maximum and global minimum of f in (a,b).

Criteria for local maxima and local minima

Let f be a differentiable function on an open interval (a,b) containing c and suppose that $f''(c)$ exists.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

NOTE

In Economics, if $y = f(x)$ represent cost function or revenue function, then the point at which $\frac{dy}{dx} = 0$, the cost or revenue is maximum or minimum.

Example 6.25

Find the extremum values of the function $f(x)=2x^3+3x^2-12x$.

Solution :

Given $f(x) = 2x^3+3x^2-12x \quad \ldots (1)$

$f'(x) = 6x^2+6x-12$

$f''(x) = 12x + 6$

$f'(x) = 0 \Rightarrow 6x^2+6x-12 = 0$

$\Rightarrow 6(x^2+x-2) = 0$

$\Rightarrow 6(x+2)(x-1) = 0$

$\Rightarrow x = -2 \; ; \; x = 1$

When $x = -2$

$f''(-2) = 12(-2) + 6$

$= -18 < 0$

$\therefore f(x)$ attains local maximum at $x = -2$ and local maximum value is obtained from (1) by substituting the value $x = -2$

$$f(-2) = 2(-2)^3+3(-2)^2-12(-2)$$
$$= -16+12+24$$
$$= 20.$$

When $x = 1$

$$f''(1) = 12(1) + 6$$
$$= 18.$$

$f(x)$ attains local minimum at $x = 1$ and the local minimum value is obtained by substituting $x = 1$ in (1).

$$f(1) = 2(1) + 3(1) - 12(1)$$
$$= -7$$

Extremum values are -7 and 20.

Example 6.26

Find the absolute (global) maximum and absolute minimum of the function

$f(x)=3x^5-25x^3+60x+1$ in the interval $[-2,2]$

Solution :

$$f(x) = 3x^5-25x^3+60x+1 \qquad \ldots (1)$$
$$f'(x) = 15x^4-75x^2+60$$
$$= 15(x^4-5x^2+4)$$
$$f'(x) = 0 \quad \Rightarrow 15(x^4-5x^2+4) = 0$$
$$\Rightarrow (x^2-4)(x^2-1) = 0$$
$$x = \pm 2 \text{ (or) } x = \pm 1$$

of these four points $-2, \pm 1 \in [-2,1]$ and $2 \notin [-2,1]$

From (1)

$$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1$$
$$= -15$$

When $x = 1$

$$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1$$
$$= 39$$

When $x = -1$

$$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1$$
$$= -37.$$

Absolute maximum is 39

Absolute minimum is -37

6.3 Applications of maxima and minima

6.3.1 Problems on profit maximization and minimization of cost function:

Example 6.27

For a particular process, the cost function is given by $C = 56 - 8x + x^2$, where C is cost per unit and x, the number of unit's produced. Find the minimum value of the cost and the corresponding number of units to be produced.

Solution :

$$C = 56 - 8x + x^2$$

Differentiate with respect to x,

$$\frac{dC}{dx} = -8 + 2x$$
$$\frac{d^2C}{dx^2} = 2$$
$$\frac{dC}{dx} = 0 \Rightarrow -8 + 2x = 0$$

$\therefore \quad x = 4$

When $x = 4$

$$\frac{d^2C}{dx^2} = 2 > 0$$

$\therefore$ C is minimum when $x = 4$.

The minimum value of cost = 56–32+16

= 40

The corresponding number of units produced = 4

Example 6.28

The total cost function of a firm is $C(x) = \frac{x^3}{3} - 5x^2 + 28x + 10$, where x is the output. A tax at the rate of ₹ 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $p = 2530 - 5x$, where p is the price per unit of output, find the profit maximizing the output and price.

Solution :

Total revenue: $R = p\,x$

$= (2530 - 5x)x$

$= 2530x - 5x^2$

Tax at the rate ₹ 2 per x unit = $2x$.

$$\therefore \quad C(x) + 2x = \frac{x^3}{3} - 5x^2 + 28x + 10 + 2x$$

$$P = \text{Total revenue} - (\text{Total cost} + \text{tax})$$

$$= (2530-5x)x - \left(\frac{x^3}{3} - 5x^2 + 28x + 10 + 2x\right)$$

$$= -\frac{x^3}{3} + 2500x - 10$$

$$\frac{dP}{dx} = -x^2 + 2500$$

$$\frac{d^2P}{dx^2} = -2x$$

$$\frac{dP}{dx} = 0 \Rightarrow 2500 - x^2 = 0$$

$$\Rightarrow \quad x^2 = 2500$$

$\therefore$ $x = 50$ (–50 is not acceptable)

At $\quad x = 50 \quad \frac{d^2P}{dx^2} = -100 < 0$

P is maximum when $x = 50$.

$$P = 2530 - 5(50)$$

$$= ₹\ 2280.$$

Example 6.29

The manufacturing cost of an item consists of ₹ 1,600 as over head material cost ₹ 30 per item and the labour cost ₹ $\left(\frac{x^2}{100}\right)$ for x items produced. Find how many items be produced to have the minimum average cost.

Solution :

As per given information for producing x units of certain item $C(x)$ = labour cost + material cost + overhead cost

$$C(x) = \frac{x^2}{100} + 30x + 1600$$

$$AC = \frac{C(x)}{x}$$

$$= \frac{\frac{x^2}{100} + 30x + 1600}{x}$$

$$= \frac{x}{100} + 30 + \frac{1600}{x}$$

$$\frac{d(AC)}{dx} = \frac{1}{100} - \frac{1600}{x^2}$$

$$\frac{d^2(AC)}{dx^2} = \frac{3200}{x^3}$$

$$\frac{d(AC)}{dx} = 0 \qquad \Rightarrow -\frac{1600}{x^2} + \frac{1}{100} = 0$$

$$\Rightarrow \frac{1}{100} = \frac{1600}{x^2}$$

$$x^2 = 160000$$

$\therefore \quad x = 400$ (–400 is not acceptable)

When $\quad x = 400, \quad \frac{d^2(AC)}{dx^2} = \frac{3200}{400^3} > 0$

AC is minimum at $x = 400$

Hence 400 items should be produced for minimum average cost.

Exercise 6.2

1. The average cost function associated with producing and marketing x units of an item is given by $AC = 2x - 11 + \frac{50}{x}$. Find the range of values of the output x, for which AC is increasing.

2. A television manufacturer finds that the total cost for the production and marketing of x number of television sets is $C(x) = 300x^2 + 4200x + 13,500$. If each product is sold for ₹ 8,400, show that the profit of the company is increasing.

3. A monopolist has a demand curve $x = 106 - 2p$ and average cost curve $AC = 5 + \frac{x}{50}$, where p is the price per unit output and x is the number of units of output. If the total revenue is $R = px$, determine the most profitable output and the maximum profit.

4. A tour operator charges ₹136 per passenger with a discount of 40 paisa for each passenger in excess of 100. The operator requires at least 100 passengers to operate the tour. Determine the number of passenger that will maximize the amount of money the tour operator receives.

5. Find the local minimum and local maximum of $y = 2x^3 - 3x^2 - 36x + 10$.

6. The total revenue function for a commodity is $R = 15x + \frac{x^2}{3} - \frac{1}{36}x^4$. Show that at the highest point average revenue is equal to the marginal revenue.

6.3.2 Inventory control

Inventory is any stored resource that is used to satisfy a current or a future need. Raw materials, finished goods are examples of inventory. The inventory problem involves placing and receiving orders of given sizes periodically so that the **total cost of inventory** is minimized.

An inventory decisions

1. How much to order ? 2. When to order ?

Costs involved in an inventory problems

(i) Holding cost or storage cost or inventory carrying cost(C_1) :

The cost associated with carrying or holding the goods in stock is known as holding cost per unit per unit time.

(ii) Shortage cost (C_2) :

The penalty costs that are incurred as a result of running out of stock are known as shortage cost.

(iii) Setup cost or ordering cost or procurement cost (C_3) :

This is the cost incurred with the placement of order or with the initial preparation of production facility such as resetting the equipment for production.

6.3.3 Economic Order Quantity(EOQ):

Economic order quantity is that size of order which minimizes total annual cost of carrying inventory and the cost of ordering under the assumed conditions of certainty with the annual demands known. Economic order quantity (EOQ) is also called Economic lot size formula.

The derivation of this formula is given for better understanding and is exempted from examination.

The formula is to determine the optimum quantity ordered (or produced) and the optimum interval between successive orders, if the demand is known and uniform with no shortages.

Let us have the following assumptions.

(i) Let R be the uniform demand per unit time.

(ii) Supply or production of items to the inventory is instantaneous.

(iii) Holding cost is ₹ C_1 per unit time.

(iv) Let there be 'n' orders (cycles) per year, each time 'q' units are ordered (produced).

(v) Let ₹ C_3 be the ordering (set up) cost per order (cycle). Let 't' be the time taken between each order.

Diagrammatic representation of this model is given below:

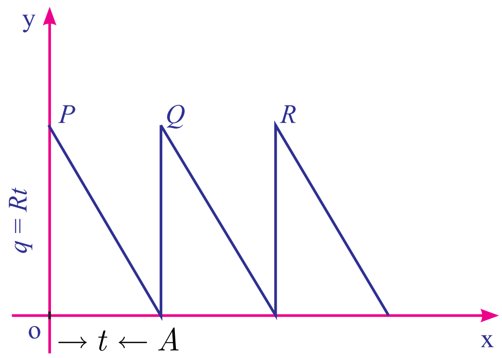


Fig. 6.12

If a production run is made at intervals t, a quantity $q = Rt$ must be produced in each run. Since the stock in small time dt is $Rt\,dt$, the stock in period t is

$$\int_0^t Rt\,dt = \frac{1}{2}Rt^2$$

$$= \frac{1}{2}qt \quad (Rt = q)$$

= Area of the inventory triangle OAP (Fig.6.12)

Cost of holding inventory per production run = $\frac{1}{2}C_1Rt^2$

Set up cost per production run = C_3.

Total cost per production run = $\frac{1}{2}C_1Rt^2 + C_3$

Average total cost per unit time $C(t) = \frac{1}{2}C_1Rt + \frac{C_3}{t}$ ... (1)

$$\frac{d}{dt}C(t) = \frac{1}{2}C_1R - \frac{C_3}{t^2} \quad \text{... (2)}$$

$$\frac{d^2C(t)}{dt^2} = \frac{2C_3}{t^3} \quad \text{... (3)}$$

$C(t)$ is minimum if $\frac{d}{dt}C(t) = 0$ and $\frac{d^2}{dt^2}C(t) > 0$

$$\frac{d}{dt}C(t) = 0 \Rightarrow \frac{1}{2}C_1R - \frac{C_3}{t^2} = 0$$

$$\Rightarrow t = \sqrt{\frac{2C_3}{C_1R}}$$

When $\sqrt{\frac{2C_3}{C_1R}}$, $\frac{d^2C(t)}{dt^2} = \frac{2C_3}{\left(\frac{2C_3}{C_1R}\right)^{\frac{3}{2}}} > 0$

Thus $C(t)$ is minimum for optimum time interval

$$t_0 = \sqrt{\frac{2C_3}{C_1 R}}$$

Optimum quantity q_0 to be produced during each production run,

$$EOQ = q_0 = Rt_0 = \sqrt{\frac{2C_3 R}{C_1}}$$

This is known as the Optimal Lot-size formula due to Wilson.

(i) Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{\text{EOQ}} = R\sqrt{\frac{C_1}{2C_3 R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

(ii) Minimum average cost per unit time, $C_0 = \sqrt{2C_1 C_3 R}$

(iii) Carrying cost $= \frac{q_0}{2} \times C_1$, Ordering cost $= \frac{R}{q_0} \times C_3$

(iv) At EOQ, Ordering cost = Carrying cost

We will discuss EOQ problems with no shortages only.

Example 6.30

A company uses 48000 units of a raw material costing ₹ 2.5 per unit. Placing each order costs ₹45 and the carrying cost is 10.8 % per year of the average inventory. Find the EOQ, total number of orders per year and time between each order. Also verify that at EOQ carrying cost is equal to ordering cost.

Solution:

Here demand rate $R = 48000$

Inventory cost $C_1 = 10.8\%$ of $2.5 = \frac{10.8}{100} \times 2.5 = 0.27$

Ordering cost $C_3 = 45$

Economic order quantity $q_0 = \sqrt{\frac{2C_3 R}{C_1}}$

$$= \sqrt{\frac{2 \times 45 \times 48000}{0.27}} = 4000 \text{ units}$$

Number of orders per year $= \frac{R}{q_0}$

$$= \frac{48000}{4000} = 12$$

$$\text{Time between orders } t_0 = \frac{q_0}{R}$$
$$= \frac{1}{12} = 0.083 \text{ year}$$
$$\text{At } EOQ\text{, carrying cost } = \frac{q_0}{2} \times C_1$$
$$= \frac{4000}{2} \times 0.27 = ₹\ 540$$
$$\text{Ordering cost} = \frac{R}{q_0} \times C_3$$
$$= \frac{48000}{4000} \times 45 = ₹\ 540$$

So at EOQ carrying cost is equal to ordering cost.

Example 6.31

A manufacturer has to supply 12,000 units of a product per year to his customer. The ordering cost (C_3) is ₹ 100 per order and carrying cost is ₹ 0.80 per item per month. Assuming there is no shortage cost and the replacement is instantaneous, determine the

(i) economic order quantity

(ii) time between orders

(iii) number of orders per year

Solution:

Demand per year : R = 12,000 units

Ordering cost: C_3= ₹ 100/order

Carrying cost : C_1= 0.80/item/month

= 0.80×12 per year

= ₹ 9.6 per year

(i) $$\text{EOQ} = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 100 \times 12000}{9.6}}$$

= 500 units

(ii) Number of order per year $= \dfrac{Demand}{EOQ} = \dfrac{12{,}000}{500} = 24$

(iii) Optimal time per order $= \dfrac{1}{t_0} = \dfrac{1}{24} \text{ year} = \dfrac{12}{24}$ months

$= \dfrac{1}{2}$ month = 15 days

Example 6.32

A company has to supply 1000 item per month at a uniform rate and for each time, a production run is started with the cost of ₹ 200. Cost of holding is ₹ 20 per item per month. The number of items to be produced per run has to be ascertained. Determine the total of setup cost and average inventory cost if the run size is 500, 600, 700, 800. Find the optimal production run size using EOQ formula.

Solution:

Demand : R = 1000 per month

Setup cost : C_3 = ₹ 200 per order

Carrying cost: C_1= ₹ 20 per item per month.

Run size q	Set up cost $\frac{R}{q} \times C_3$	Carrying cost $\frac{q}{2} \times C_1$	Total cost (Set up cost + Carrying cost)
500	$\frac{1000}{500} \times 200 = 400$	$\frac{500}{2} \times 20 = 5000$	5400
600	$\frac{1000}{600} \times 200 = 333.3$	$\frac{600}{2} \times 20 = 6000$	6333.3
700	$\frac{1000}{700} \times 200 = 285.7$	$\frac{700}{2} \times 20 = 7000$	7285.7
800	$\frac{1000}{800} \times 200 = 250$	$\frac{800}{2} \times 20 = 8000$	8250

Table : 6.3

$$EOQ = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 1000 \times 200}{20}}$$

$$= \sqrt{20000}$$

$$= 141 \text{ units (app.)}$$

Example 6.33

A manufacturing company has a contract to supply 4000 units of an item per year at uniform rate. The storage cost per unit per year amounts to ₹ 50 and the set up cost per production run is ₹ 160. If the production run can be started instantaneously and shortages are not permitted, determine the number of units which should be produced per run to minimize the total cost.

Solution :

Annual demand : $R = 4000$

Storage cost: $C_1 = ₹\ 50$

Setup cost per production: $C_3 = ₹160$

$$EOQ = \sqrt{\frac{2Rc_3}{c_1}}$$

$$= \sqrt{\frac{2 \times 4000 \times 160}{50}} = 160.$$

$\therefore$ To minimize the production cost number of units produced per run is 160 units.

Example 6.34

A company buys in lots of 500 boxes which is a 3 month supply. The cost per box is ₹125 and the ordering cost in ₹150. The inventory carrying cost is estimated at 20% of unit value.

(i) Determine the total amount cost of existing inventory policy

(ii) How much money could be saved by applying the economic order quantity?

Solution:

Given

Ordering cost per order : $C_3 = ₹150$ per order.

Number of units per order: $q = 500$ units

Annual demand $= 500 \times 4 = 2000$ units

$\therefore$ Demand rate : $R = 2000$ per year

Carrying cost : $C_1 = 20\%$ of unit value

$$C_1 = \frac{20}{100} \times 125 = ₹25$$

(i) Total annual cost of due existing inventory policy

$$= \frac{R}{q} \times C_3 + \frac{q}{2} C_1$$

$$= \frac{2000}{500} \times 150 + \frac{500}{2} \times 25$$

$$= ₹\ 6850$$

(ii) $$EOQ = \sqrt{\frac{2RC_3}{C_1}}$$

$$= \sqrt{\frac{2 \times 2000 \times 150}{25}}$$

$$= \sqrt{12 \times 2000}$$

$$= 155 \text{ (app.)}$$

Minimum annual cost $= \sqrt{2RC_3C_1}$

$$= \sqrt{2 \times 2000 \times 150 \times 25}$$

$$= ₹3873.$$

By applying the economic order quantity, money saved by a company = 6850−3873

= ₹2977.

Exercise 6.3

1. The following table gives the annual demand and unit price of 3 items

Items	Annual Demand (units)	Unit Price ₹
A	800	0.02
B	400	1.00
C	13,800	0.20

Ordering cost is Rs. 5 per order and holding cost is 10% of unit price.

Determine the following:

(i) EOQ in units

(ii) Minimum average cost

(iii) EOQ in rupees

(iv) EOQ in years of supply

(v) Number of orders per year.

2. A dealer has to supply his customer with 400 units of a product per every week. The dealer gets the product from the manufacturer at a cost of ₹ 50 per unit. The cost of ordering from the manufacturers in ₹ 75 per order. The cost of holding inventory is 7.5 % per year of the product cost. Find (i) EOQ (ii) Total optimum cost.

6.4 Partial Derivatives

Partial derivative of a function of several variables is its derivative with respect to one of those variables, keeping other variables as constant. In this section, we will restrict our study to functions of two variables and their derivatives only.

Let $u = f(x, y)$ be a function of two independent variables x and y.

The derivative of u with respect to x when x varies and y remains constant is called the **partial derivative of u with respect to x,** denoted by $\frac{\partial u}{\partial x}$ (or) u_x and is defined as

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

provided the limit exists. Here Δx is a small change in x

The derivative of u with respect to y, when y varies and x remains constant is called the **partial derivative of u with respect to y,** denoted by $\frac{\partial u}{\partial y}$ (or) u_y and is defined as

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \longrightarrow 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

provided the limit exists. Here Δy is a small change in y.

$\frac{\partial u}{\partial x}$ is also written as $\frac{\partial}{\partial x} f(x,y)$ (or) $\frac{\partial f}{\partial x}$.

Similarly $\frac{\partial u}{\partial y}$ is also written as $\frac{\partial}{\partial y} f(x,y)$ (or) $\frac{\partial f}{\partial y}$

The process of finding a partial derivative is called **partial differentiation.**

6.4.1 Successive partial derivatives

Consider the function $u = f(x, y)$. From this we can find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. If $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of x and y, then they may be differentiated partially again with respect to either of the independent variables, (x or y) denoted by $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y \partial x}$.

These derivatives are called second order partial derivatives. Similarly, we can find the third order partial derivatives, fourth order partial derivatives etc., if they exist. The process of finding such partial derivatives are called **successive partial derivatives.**

If we differentiate $u = f(x,y)$ partially with respect to x and again differentiating partially with respect to y, we obtain $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ i.e., $\frac{\partial^2 u}{\partial y \partial x}$

Similarly, if we differentiate $u = f(x,y)$ partially with respect to y and again differentiating partially with respect to x, we obtain $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ i.e., $\frac{\partial^2 u}{\partial x \partial y}$

NOTE

If $u(x,y)$ is a continuous function of x and y then $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

Homogeneous functions

A function $f(x,y)$ of two independent variables x and y is said to be homogeneous in x and y of degree n if $f(tx,ty) = t^n f(x,y)$ for $t > 0$.

6.4.2 Euler's theorem and its applications

Euler's theorem for two variables:

If $u = f(x,y)$ is a homogeneous function of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Example 6.35

If $u = x^2(y-x) + y^2(x-y)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2(x-y)^2$.

Solution:

$$u = x^2y - x^3 + xy^2 - y^3$$

$$\frac{\partial u}{\partial x} = 2xy - 3x^2 + y^2$$

$$\frac{\partial u}{\partial y} = x^2 + 2xy - 3y^2$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2x^2 - 2y^2 + 4xy$$

$$= -2\left(x^2 + y^2 - 2xy\right)$$

$$= -2(x-y)^2$$

Example 6.36

If $u = \log(x^2+y^2)$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution:

$$u = \log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right)\cdot 2 - 2x\cdot 2x}{\left(x^2+y^2\right)^2} = \frac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Example 6.37

If $u = xy+\sin(xy)$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution:

$u = xy+\sin(xy)$

$$\frac{\partial u}{\partial x} = y+y\cos(xy)$$

$$\frac{\partial u}{\partial y} = x+x\cos(xy)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= 1 + x\left(-\sin\left(xy\right)\cdot y\right) + \cos\left(xy\right)$$

$$= 1-xy\sin(xy)+\cos(xy) \ldots. \quad (1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(y + y\cos\left(xy\right)\right)$$

$$= 1 + \cos\left(xy\right) + y\left(-\sin\left(xy\right)x\right)$$

$$= 1 - xy\sin\left(xy\right) + \cos\left(xy\right) \ldots \ (2)$$

From (1) and (2), we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

Example 6.38

Verify Euler's theorem for the function $u = \frac{1}{\sqrt{x^2+y^2}}$.

Solution:

$$u(x,y) = \left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$u(tx,ty) = \left(t^2x^2+t^2y^2\right)^{-\frac{1}{2}} = t^{-1}\left(x^2+y^2\right)^{-\frac{1}{2}}$$

$\therefore$ u is a homogeneous function of degree -1

By Euler's theorem , $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = (-1)u = -u$

Verification:

$$u = \left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\cdot 2x = \frac{-x}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$x\cdot\frac{\partial u}{\partial x} = \frac{-x^2}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\cdot 2y = \frac{-y}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$y\cdot\frac{\partial u}{\partial y} = \frac{-y^2}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$\therefore \quad x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{-(x^2+y^2)}{(x^2+y^2)^{-\frac{3}{2}}}$$

$$= (-1)\frac{1}{\sqrt{x^2+y^2}} = (-1)u = -u$$

Hence Euler's theorem verified.

Example 6.39

Let $u = \log\dfrac{x^4+y^4}{x+y}$. By using Euler's theorem show that $x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} = 3$.

Solution:

$$u = \log\frac{x^4+y^4}{x+y}$$

$$e^u = \frac{x^4+y^4}{x+y} = f(x,y) \qquad \ldots (1)$$

Consider $\quad f(x,y) = \dfrac{x^4+y^4}{x+y}$

$$f(tx,ty) = \frac{t^4x^4+t^4y^4}{tx+ty} = t^3\left(\frac{x^4+y^4}{x+y}\right) = t^3 f(x,y)$$

$\therefore$ f is a homogeneous function of degree 3.

Using Euler's theorem we get, $x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} = 3f$

Consider $f(x,y) = e^u$

$$x \cdot \frac{\partial e^u}{\partial x} + y \cdot \frac{\partial e^u}{\partial y} = 3e^u$$

$$\therefore \quad e^u x \cdot \frac{\partial u}{\partial x} + e^u y \cdot \frac{\partial u}{\partial y} = 3e^u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3$$

Exercise 6.4

1. If $z = (ax+b)(cy+d)$, then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
2. If $u = e^x y$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u(x^2+y^2)$
3. Let $u = x\cos y + y\cos x$. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$
4. Verify Euler's theorem for the function $u = x^3 + y^3 + 3xy^2$
5. Let $u = x^2 y^3 \cos\left(\frac{x}{y}\right)$. By using Euler's theorem show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 5u$.

6.5. Applications of partial derivatives

In this section we solve problems on partial derivatives which have direct impact on Industrial areas.

6.5.1 Production function and marginal productivities of two variables

(i) Production function:

Production P of a firm depends upon several economic factors like capital (K), labour (L), raw materials (R), machinery (M) etc... Thus $P = f(K, L, R, M, ...)$ is known as production function. If P depends only on labour (L) and capital (K), then we write $P = f(L, K)$.

(ii) Marginal productivities:

Let $P = f(L, K)$ be a production function. Then $\frac{\partial P}{\partial L}$ is called the **Marginal productivity of labour** and $\frac{\partial P}{\partial K}$ is called the **Marginal productivity of capital**.

Euler's theorem for homogeneous production function $P(L, K)$ of degree 1 states that

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P$$

6.5.2 Partial elasticity of demand

Let $q = f(p_1, p_2)$ be the demand for commodity A, which depends upon the prices p_1 and p_2 of commodities A and B respectively.

The partial elasticity of demand q with respect to p_1 is defined to be

$$\eta_{qp_1} = \frac{Eq}{Ep_1} = \frac{-p_1}{q}\frac{\partial q}{\partial p_1}$$

The partial elasticity of demand q with respect to p_2 is defined to be

$$\eta_{qp_2} = \frac{Eq}{Ep_2} = \frac{-p_2}{q}\frac{\partial q}{\partial p_2}$$

Example 6.40

Find the marginal productivities of capital (K) and labour (L) if

$P = 10L + 0.1L^2 + 5K - 0.3K^2 + 4KL$ when K=L=10.

Solution:

We have $P = 10L + 0.1L^2 + 5K - 0.3K^2 + 4KL$

$$\frac{\partial P}{\partial L} = 10 + 0.2L + 4K$$

$$\frac{\partial P}{\partial K} = 5 - 0.6K + 4L$$

Marginal productivity of labour: $\left(\frac{\partial P}{\partial L}\right)_{(10,10)} = 10+2+40=52$

Marginal productivity of capital: $\left(\frac{\partial P}{\partial K}\right)_{(10,10)} = 5-6+40=39$

Example 6.41

The production function for a commodity is $P=10L + 0.1L^2 + 15K - 0.2K^2 + 2KL$ where L is labour and K is Capital.

(i) Calculate the marginal products of two inputs when 10 units of each of labour and Capital are used

(ii) If 10 units of capital are used, what is the upper limit for use of labour which a rational producer will never exceed?

Solution:

(i) Given the production is $P = 10L - 0.1L^2 + 15K - 0.2K^2 + 2KL$

$$\frac{\partial P}{\partial L} = 10 - 0.2L + 2K$$

$$\frac{\partial P}{\partial K} = 15 - 0.4K + 2L$$

When $L = K = 10$ units,

Marginal productivity of labour : $\left(\frac{\partial P}{\partial L}\right)_{(10,10)} = 10 - 2+20=28$

Marginal productivity of capital : $\left(\frac{\partial P}{\partial K}\right)_{(10,10)} = 15-4+20=31$

(ii) Upper limit for use of labour when $K=10$ is given by $\left(\frac{\partial P}{\partial L}\right) \geq 0$

$$10 - 0.2L + 20 \geq 0$$

$$30 \geq 0.2L$$

i.e., $$L \leq 150$$

Hence the upper limit for the use of labour will be 150 units

Example 6.42

For the production function, $P = 4L^{\frac{3}{4}} K^{\frac{1}{4}}$ verify Euler's theorem.

Solution:

$P = 4L^{\frac{3}{4}} K^{\frac{1}{4}}$ is a homogeneous function of degree 1.

Marginal productivity of labour is

$$\frac{\partial P}{\partial L} = 4 \times \frac{3}{4} L^{\frac{-1}{4}} K^{\frac{1}{4}} = 3\left(\frac{K}{L}\right)^{\frac{1}{4}}$$

Marginal productivity of capital is

$$\frac{\partial P}{\partial K} = 4\, L^{\frac{3}{4}} \times \frac{1}{4} K^{\frac{-3}{4}} = \left(\frac{L}{K}\right)^{\frac{3}{4}}$$

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = 3\mathrm{L}\left(\frac{K}{L}\right)^{\frac{1}{4}} + \mathrm{K}\left(\frac{L}{K}\right)^{\frac{3}{4}}$$

$$= 3\, L^{\frac{3}{4}} K^{\frac{1}{4}} + L^{\frac{3}{4}} K^{\frac{1}{4}}$$

$$= 4\, L^{\frac{3}{4}} K^{\frac{1}{4}} = P$$

Hence Euler's theorem is verified.

Example 6.43

The demand for a commodity x is $q=5-2p_1+p_2-p_1^2p_2$. Find the partial elasticities $\frac{Eq}{EP_1}$ and $\frac{Eq}{EP_2}$ when $p_1=3$ and $p_2=7$

Solution:

$$\frac{\partial q}{\partial p_1} = -2-2p_1\,p_2$$

$$\frac{\partial q}{\partial p_2} = 1-p_1^2$$

(i)
$$\frac{Eq}{Ep_1} = -\frac{P_1}{q}\frac{\partial q}{\partial p_1} = \frac{-p_1}{5-2p_1+p_2-p_1^2p_2}(-2-2p_1p_2)$$

$$= \frac{2p_1+2p_1^2p_2}{5-2p_1+p_2-p_1^2p_2}$$

When $p_1=3$ and $p_2=7$

$$\frac{Eq}{Ep_1} = \frac{2(3)+2(9)(7)}{5-6+7-(9)(7)} = \frac{132}{-57} = \frac{-132}{57}$$

(ii)
$$\frac{Eq}{Ep_2} = -\frac{P_2}{q}\frac{\partial q}{\partial p_2} = \frac{-p_2(1-p_1^2)}{5-2p_1+p_2-p_1^2p_2}$$

$$= \frac{-p_2+p_2p_1^2}{5-2p_1+p_2-p_1^2p_2}$$

When $p_1=3$ and $p_2=7$

$$\frac{Eq}{Ep_2} = \frac{-7+7(9)}{5-6+7-(9)(7)} = \frac{56}{-57} = \frac{-56}{57}$$

Exercise 6.5

1. Find the marginal productivities of capital (K) and labour (L) if $P=8L-2K+3K^2-2L^2+7KL$ when $K=3$ and $L=1$

2. If the production of a firm is given by $P=4LK-L^2+K^2$, $L>0$, $K>0$, Prove that $L\frac{\partial P}{\partial L}+K\frac{\partial P}{\partial K}=2P$

3. If the production function is $z=3x^2-4xy+3y^2$ where x is the labour and y is the capital, find the marginal productivities of x and y when $x=1$, $y=2$.

4. For the production function $P = 3(L)^{0.4}(K)^{0.6}$, find the marginal productivities of labour (L) and capital (K) when $L = 10$ and $K = 6$. [use: $(0.6)^{0.6} = 0.736, (1.67)^{0.4} = 1.2267$]

5. The demand for a quantity A is $q = 13 - 2p_1 - 3p_2^2$. Find the partial elasticities $\frac{Eq}{Ep_1}$ and $\frac{Eq}{Ep_2}$ when $p_1 = p_2 = 2$.

6. The demand for a commodity A is $q = 80 - p_1^2 + 5p_2 - p_1 p_2$. Find the partial elasticities $\frac{Eq}{Ep_1}$ and $\frac{Eq}{Ep_2}$ when $p_1 = 2$, $p_2 = 1$.

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Exercise 6.6

Choose the Correct answer:

1. Average fixed cost of the cost function $C(x) = 2x^3 + 5x^2 - 14x + 21$ is

   (a) $\frac{2}{3}$ (b) $\frac{5}{x}$ (c) $-\frac{14}{x}$ (d) $\frac{21}{x}$

2. Marginal revenue of the demand function $p = 20 - 3x$ is

   (a) $20 - 6x$ (b) $20 - 3x$ (c) $20 + 6x$ (d) $20 + 3x$

3. If demand and the cost function of a firm are $p = 2 - x$ and $C = -2x^2 + 2x + 7$ then its profit function is

   (a) $x^2 + 7$ (b) $x^2 - 7$ (c) $-x^2 + 7$ (d) $-x^2 - 7$

4. If the demand function is said to be inelastic, then

   (a) $|\eta_d| > 1$ (b) $|\eta_d| = 1$ (c) $|\eta_d| < 1$ (d) $|\eta_d| = 0$

5. The elasticity of demand for the demand function $x = \frac{1}{p}$ is

   (a) 0 (b) 1 (c) $-\frac{1}{p}$ (d) ∞

6. Relationship among MR, AR and η_d is

   (a) $\eta_d = \frac{AR}{AR - MR}$ (b) $\eta_d = AR - MR$ (c) $MR = AR = \eta_d$ (d) $AR = \frac{MR}{\eta_d}$

7. For the cost function $C = \frac{1}{25}e^{5x}$, the marginal cost is

   (a) $\frac{1}{25}$ (b) $\frac{1}{5}e^{5x}$ (c) $\frac{1}{125}e^{5x}$ (d) $25e^{5x}$

8. Instantaneous rate of change of $y = 2x^2 + 5x$ with respect to x at $x = 2$ is

(a) 4 (b) 5 (c) 13 (d) 9

9. If the average revenue of a certain firm is ₹ 50 and its elasticity of demand is 2, then their marginal revenue is

(a) ₹ 50 (b) ₹ 25 (c) ₹ 100 (d) ₹ 75

10. Profit $P(x)$ is maximum when

(a) $MR = MC$ (b) $MR = 0$ (c) $MC = AC$ (d) $TR = AC$

11. The maximum value of $f(x) = \sin x$ is

(a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

12. If $f(x,y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is equal to

(a) $(n-1)f$ (b) $n(n-1)f$ (c) nf (d) f

13. If $u = 4x^2 + 4xy + y^2 + 4x + 32y + 16$. then $\frac{\partial^2 u}{\partial y \partial x}$ is equal to

(a) $8x + 4y + 4$ (b) 4 (c) $2y + 32$ (d) 0

14. If $u = x^3 + 3xy^2 + y^3$ then $\frac{\partial^2 u}{\partial y \partial x}$ is

(a) 3 (b) $6y$ (c) $6x$ (d) 2

15. If $u = e^{x^2}$ then $\frac{\partial u}{\partial x}$ is equal to

(a) $2xe^{x^2}$ (b) e^{x^2} (c) $2e^{x^2}$ (d) 0

16. Average cost is minimum when

(a) Marginal cost = marginal revenue

(b) Average cost = marginal cost

(c) Average cost = Marginal revenue

(d) Average Revenue = Marginal cost

17. A company begins to earn profit at

(a) Maximum point (b) Breakeven point

(c) Stationary point (d) Even point

18. The demand function is always

(a) Increasing function (b) Decreasing function

c) Non-decreasing function (d) Undefined function

19. If $q = 1000 + 8p_1 - p_2$ then, $\dfrac{\partial q}{\partial p_1}$ is

(a) -1 (b) 8 (c) 1000 (d) $1000 - p_2$

20. If R = 5000 units / year, C_1= 20 paise, C_3= ₹ 20 then EOQ is

(a)5000 (b) 100 (c) 1000 (d) 200

Miscellaneous Problems

1. The total cost function for the production of x units of an item is given by $C = 10 - 4x^3 + 3x^4$ find the (i) average cost, (ii) marginal cost (iii) marginal average cost.

2. Find out the elasticity of demand for the following functions (i) $p = xe^x$ (ii) $p = xe^{-x}$ (iii) $p = 10e^{-\frac{x}{3}}$

3. Find the elasticity of supply when the supply function is given by $x = 2p^2 + 5$ at p=1

4. For the demand function $p = 100 - 6x^2$, find the marginal revenue and also show that $MR = p\left[1 - \dfrac{1}{\eta_d}\right]$

5. The total cost function y for x units is given by $y = 4x\left(\dfrac{x+2}{x+1}\right) + 6$. Prove that marginal cost decreases as x increases.

6. For the cost function C = $2000 + 1800x - 75x^2 + x^3$ find when the total cost (C) is increasing and when it is decreasing.

7. A certain manufacturing concern has total cost function C = $15 + 9x - 6x^2 + x^3$. Find x, when the total cost is minimum.

8. Let $u = \log\dfrac{x^4 - y^4}{x - y}$. Using Euler's theorem show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

9. Verify $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ for $u = x^3 + 3x^2y^2 + y^3$

10. If $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 7$ then show that $f_{yy}(1,1) = 18$.

Summary

- Demand is the relationship between the quantity demanded and the price of a commodity.
- Supply is the relationship between the quantity supplied and the price of a commodity.
- Cost is the amount spent on the production of a commodity.
- Revenue is the amount realised by selling the output produced on commodity.
- Profit is the excess of total revenue over the cost of production.
- Elasticity of a function $y = f(x)$ at a point x is the limiting case of ratio of the relative change in y to the relative change in x
- Equilibrium price is the price at which the demand of a commodity is equal to its supply.
- Marginal cost is interpreted as the approximate change in production cost of $(x+1)^{th}$ unit, when the production level is x units.
- Marginal Revenue is interpreted as the approximate change in revenue made on by selling of $(x+1)^{th}$ unit, when the sale level is x units.
- A function $f(x)$ is said to be increasing function in the interval $[a, b]$ if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in [a,b]$
- A function $f(x)$ is said to be strictly increasing in $[a, b]$ if
  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in [a,b]$
- A function $f(x)$ is said to be decreasing function in $[a,b]$ if
  $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in [a,b]$
- A function $f(x)$ is said to be strictly decreasing function in $[a, b]$if
  $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in [a,b]$
- Let f be a differentiable function on an open interval (a,b) containing c and suppose that $f'(c)$ exists.

  (i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

  (ii) If $f'(c) = 0$ and $f''(c) < 0$,then f has a local maximum at c.
- A function $f(x, y)$ of two independent variables x and y is said to be homogeneous in x and y of degree n if for $t > 0$ $f(tx,ty) = t^n f(x,y)$

- If $u = f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$
- The partial elasticity of demand q with respect to p_1 is defined to be $\eta_{qp_1} = \frac{Eq}{Ep_1} = \frac{-p_1}{q}\frac{\partial q}{\partial p_1}$
- The partial elasticity of demand q with respect to p_2 is defined to be

$$\eta_{qp_2} = \frac{Eq}{Ep_2} = \frac{-p_2}{q}\frac{\partial q}{\partial p_2}$$

Some standard results

1. Total cost: $C(x) = f(x) + k$
2. Average cost: $AC = \frac{f(x)+k}{x} = \frac{c(x)}{x}$
3. Average variable cost: AVC$= \frac{f(x)}{x}$
4. Average fixed cost: $AFC = \frac{k}{x}$
5. Marginal cost: $MC = \frac{dC}{dx}$
6. Marginal Average cost: $MAC = \frac{d}{dx}(AC)$
7. Total cost: $C\left(x\right) = AC \times x$
8. Revenue: $R = px$
9. Average Revenue: $AR = \frac{R}{x} = p$
10. Marginal Revenue: $MR = \frac{dR}{dx}$
11. Profit: $P(x) = R(x) - C(x)$
12. Elasticity: $\eta = \frac{x}{y} \cdot \frac{dy}{dx}$
13. Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

14. Elasticity of supply: $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$

15. Relationship between MR, AR and η_d : $MR = AR\left[1 - \frac{1}{\eta_d}\right]$ (or) $\eta_d = \frac{AR}{AR - MR}$

16. Marginal function of y with respect to x (or) Instantaneous rate of change of y with respect to x is $\frac{dy}{dx}$

17. Average cost [AC] is minimum when $MC = AC$

18. Total revenue [TR] is maximum when $MR = 0$

19. Profit [$P(x)$] is maximum when $MR = MC$

20. In price elasticity of a function,

 (a) If $|\eta| > 1$, then the function is elastic

 (b) If $|\eta| = 1$, then the function is unit elastic

 (c) If $|\eta| < 1$, then the function is inelastic.

21. $EOQ = q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}}$

22. Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{\text{EOQ}} = R\sqrt{\frac{C_1}{2C_3R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

23. Minimum average cost per unit time, $C_0 = \sqrt{2C_1C_3R}$

24. Carrying cost $= \frac{q_0}{2} \times C_1$ and ordering cost $= \frac{R}{q_0} \times C_3$

25. At EOQ, Ordering cost = Carrying cost

26. If $u(x,y)$ is a continuous function of x and y then, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

GLOSSARY

Approximately	தோராயமான
Average	சராசரி
Commodity	பொருள்
Consumer	நுகர்வோர்
Cost function	செலவுச் சார்பு
Demand	தேவை
Elasticity	நெகிழ்ச்சி
Equilibrium	சமநிலை
Excess	மிகுதியான
Fixed cost	ஒரே விலை / மாறா விலை
Marginal	இறுதிநிலை / விளிம்பு
Maximum	பெருமம்
Minimum	சிறுமம்
Production Output	உற்பத்தி வெளியீடு
Producer	உற்பத்தியாளர்
Profit	இலாபம்
Quantity	அளவு
Rate of change	மாறுவீதம்
Ratio	விகிதம்
Relative change	சார்ந்த மாற்றம்
Revenue function	வருவாய் சார்பு
Supply	அளிப்பு
Variable cost	மாறும் விலை

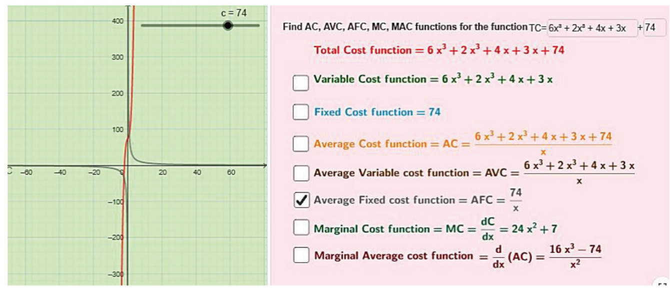
ICT Corner

Application of Differentiation

Step – 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "11th Business Maths Volume-2" will appear. In that there are several worksheets related to your Text Book.

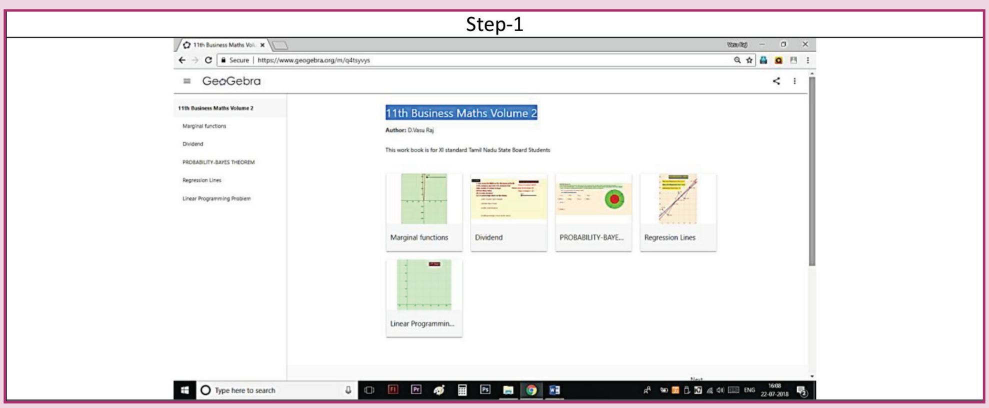


Expected Outcome ⇒

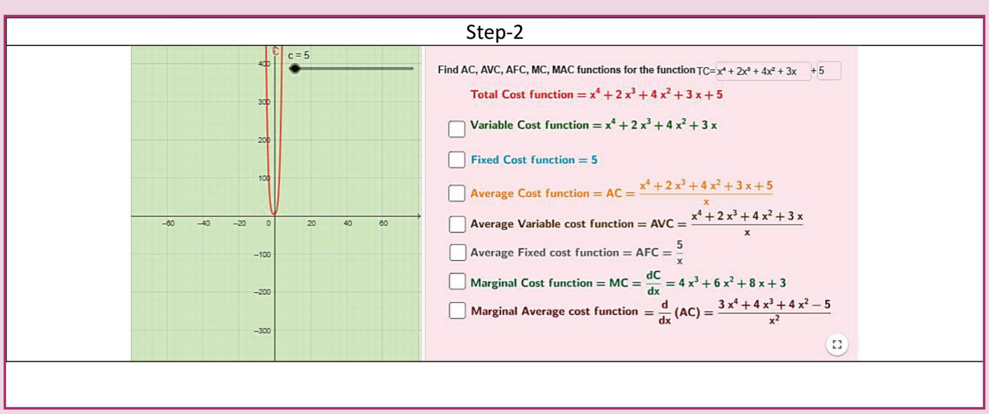
Step-2

Select the work sheet "Marginal function" Click on respective boxes to see the graph on left side. Type the total cost function in the box seen on right side and proceed.

Step 1



Step 2



Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

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Chapter 7

Financial Mathematics

Learning objectives

After studying this chapter, the students will be able to understand

- Annuities - types of annuities
- Future and present values of annuities
- The concept of gain or loss in the sale and purchase of a stock.
- Brokerage in share transactions
- Effective rate of return.

IJHW3V

INTRODUCTION

In our day-to-day life we have lot of **money** transactions. In many transactions payment is made in single payment or in equal installments over a certain period of time. The amounts of these installments are determined in such a way that they compensate for their waiting time. In other cases, in order to meet future planned expenses, a regular saving may be done. (i.e) at regular time intervals, a certain amount may be kept aside, on which the person gains interest. In such cases the concept of annuity is used.

7.1 Annuities

A sequence of equal payments made/received at equal intervals of time is called **annuity**. The amount of regular payment of an annuity is called **periodic payment**. The time interval between two successive payments is called **payment interval** or **payment period**. Note that, the payment period may be annual, half yearly, quarterly, monthly (or) any fixed duration of time. The time interval between the first payment and the last payment of an annuity is called **term of an annuity**.

The sum of all payments made and interest earned on them at the end of the term of annuities is called future value of an annuity. The present or capital value of an annuity is the sum of the present values of all the payments of the annuity at the beginning of the annuity of purchase the payments due in future. Here we note that unless mentioned specifically, the payment means yearly payment.

7.1.1 Types of annuities

a) Based on the number of periods:

(i) Certain annuity: An annuity payable for a fixed number of years is called certain annuity.

Installments of payment for a plot of land, Bank security deposits, purchase of domestic durables are examples of certain annuity. Here the buyer knows the specified dates on which installments are to be made.

(ii) Annuity contingent: An annuity payable at regular interval of time till the happening of a specific event or the date of which cannot be accurately foretold is called annuity contingent.

For example the endowment funds of trust, where the interest earned is used for welfare activities only. The principal remains the same and activity continues forever.

All the above types of annuities are based on the number of their periods. An annuity can also be classified on the basis of mode of payment as under.

b) Based on the mode of payment :

i) Ordinary annuity: An annuity in which payments of installments are made at the end of each period is called ordinary annuity (or immediate annuity)

For example repayment of housing loan, vehicle loan etc.,

ii) Annuity due: An annuity in which payments of installments are made in the beginning of each period is called annuity due.

In annuity due every payment is an investment and earns interest. Next payment will earn interest for one period less and so on, the last payment will earn interest of one period.

For example saving schemes, life insurance payments, etc.,

DO YOU KNOW?

Deferred annuity: An annuity which is payable after the lapse of a number of periods is called deferred annuity.

The derivation of the following formulae are given for better understanding and are exempted from examination

(i) Amount of immediate annuity (or) Ordinary annuity (or) Certain annuity

Let 'a' be the ordinary annuity and i percent be the rate of interest per period. In ordinary annuity, the first installment is paid after the end of first period. Therefore it earns interest for $(n - 1)$ period, second installment earns interest for $(n - 2)$ periods and so on. The last installment earns for $(n - n)$ periods. (i.e) earns no interest.

For $(n-1)$ periods,

The amount of first annuity $= a(1+i)^{n-1}$

The amount of second annuity $= a(1+i)^{n-2}$

The amount of third annuity $= a(1+i)^{n-3}$ and so on.

$\therefore$ The total amount of annuity A for n period at i percent rate of interest is

$$A = a(1+i)^{n-1}+a(1+i)^{n-2}+\ldots a(1+i)+a$$

$$= a[(1+i)^{n-1}+(1+i)^{n-2}+\ldots+(1+i)+1]$$

$$= a[1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}]$$

$$= a[1+r+r^2+\ldots+r^{n-1}], \text{ where } 1+i=r$$

$$= a\left[\frac{r^n-1}{r-1}\right], \quad \text{G.P with common ratio } r>1$$

$$= a\left[\frac{(1+i)^n-1}{1+i-1}\right]$$

$$\boldsymbol{A = \frac{a}{i}\left[(1+i)^n-1\right]}$$

(ii) Present Value of immediate annuity (or ordinary annuity)

Let 'a' be the annual payment of an ordinary annuity, n be the number of years and i percent be the interest on one rupee per year and P be the present value of the annuity. In the case of immediate annuity, payments are made periodically at the end of specified period. Since the first installment is paid at the end of first year, its present value is $\frac{a}{1+i}$, the present value of second installment is $\frac{a}{(1+i)^2}$ and so on. If the present value of last installment is $\frac{a}{(1+i)^n}$, then we have

$$P = \frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots\frac{a}{(1+i)^n}$$

$$= \frac{a}{(1+i)^n}+\frac{a}{(1+i)^{n-1}}+\ldots+\frac{a}{(1+i)}$$

$$= \frac{a}{r^n}[1+r+r^2+\ldots+r^{n-1}], \text{ taking } 1+i=r$$

$$= \frac{a}{r^n}\left[\frac{r^n-1}{r-1}\right] \text{ , G.P with common ratio, } r > 1$$

$$= \frac{a}{(1+i)^n}\left[\frac{(1+i)^n-1}{1+i-1}\right]$$

$$= \frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$\boldsymbol{P = \frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]}$$

(iii) Amount of annuity due at the end of *n* period

Annuity due is an annuity in which the payments are made at the beginning of each payment period. The first installment will earn interest for n periods at the rate of 'i' percent per period. Similarly second installment will earn interest for $(n - 1)$ periods, and so on the last interest for on period.

$$\therefore A = a(1+i)^n+a(1+i)^{n-1}+\ldots+a(1+i)^1$$

$$= a(1+i)[(1+i)^{n-1}+(1+i)^{n-2})+\ldots+1]$$

$$= a(1+i)[1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-1}]$$

$$= ar[1+r+r^2+\ldots+r^{n-1}] \text{ , put } 1+i = r$$

$$= ar[\frac{r^n-1}{r-1}] \text{ , G.P with common ratio, } r > 1$$

$$= a(1+i)\left[\frac{(1+i)^n-1}{1+i-1}\right]$$

$$= \frac{a(1+i)}{i}[(1+i)^n-1]$$

$$\boldsymbol{A = \frac{a(1+i)}{i}[(1+i)^n-1]}$$

(iv) Present value of annuity due

Since the first installment is paid at the beginning of the first period (year), its present value will be the same as 'a', the annual payment of annuity due. The second installment is paid in the beginning of the second year, hence its present value is given by $\frac{a}{(1+i)}$ and so on. The last installment is paid in the beginning of nth year, hence its present value is given as $\frac{a}{(1+i)^{n-1}}$

If P denotes the present value of annuity due, then

$$P = a+\frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots+\frac{a}{(1+i)^{n-1}}$$

$$= a\left[1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\ldots+\frac{1}{(1+i)^{n-1}}\right]$$

$$= \quad a\left[\frac{1-\left(\frac{1}{1+i}\right)^{n}}{1-\frac{1}{1+i}}\right]$$

$$= \quad a\left[\frac{\frac{(1+i)^{n}-1}{(1+i)^{n}}}{\frac{(1+i)-1}{1+i}}\right]$$

$$= \quad \frac{a(1+i)}{i}\left[\frac{(1+i)^{n}-1}{(1+i)^{n}}\right]$$

$$P \quad = \quad \boldsymbol{\frac{a(1+i)}{i}}\left[1-\frac{1}{(1+i)^{n}}\right]$$

(v) Perpetual Annuity

Perpetual annuity is an annuity whose payment continuous for ever. As such the amount of perpetuity is undefined as the amount increases without any limit as time passes on. We know that the present value P of immediate annuity is given by

$$P \quad = \quad \frac{a}{i}\left[1-\frac{1}{(1+i)^{n}}\right]$$

Now as per the definition of perpetual annuity as $n \to \infty$, we know that $\frac{1}{(1+i)^{n}} \to 0$ since $1+i > 1$.

Here $P \quad = \quad \frac{a}{i}[1-0]$

$$P \quad = \quad \frac{a}{i}$$

NOTE

In all the above formulae the period is of one year. Now if the payment is made more than once in a year then 'i' is replaced by $\frac{i}{k}$ and n is replaced by nk, where k is the number of payments in a year.

Example 7.1

A person pays ₹ 64,000 per annum for 12 years at the rate of 10% per year. Find the annuity $[(1.1)^{12}=3.3184]$

Solution :

Here a = 64,000, n = 12 and $i = \frac{10}{100} = 0.1$

Ordinary annuity $A \quad = \quad \frac{a}{i}[(1+i)^{n}-1]$

$$= \frac{64000}{0.1}[(1+0.1)^{12}-1]$$

$$= 6,40,000[(1.1)^{12}-1]$$

$$= 6,40,000[3.3184 - 1]$$

$$= 6,40,000\ [2.3184]$$

$$= 64\times23184$$

$$= 14,83,776$$

Example 7.2

What amount should be deposited annually so that after 16 years a person receives ₹1,67,160 if the interest rate is 15% $[(1.15)^{16}=9.358]$

Solution :

Here $A = 1,67,160$, $n = 16$ $i=\frac{15}{100}=0.15$ $a = ?$

To find : a

$$\text{Now} \quad A = \frac{a}{i}[(1+i)^n-1]$$

$$1,67,160 = \frac{a}{0.15}[(1+0.15)^{16}-1]$$

$$= \frac{a}{0.15}[(1.15)^{16}-1]$$

$$a = \frac{1,67,160\times0.15}{(1.15)^{16}-1}$$

$$a = \frac{1,67,160\times0.15}{9.358-1}$$

$$= \frac{1,67,160\times0.15}{8.358}$$

$$= 3,000$$

$$\text{Therefore } a = 3,000$$

Example 7.3

The age of the daughter is 2 years. Her father wants to get ₹20,00,000 when his ward becomes 22 years He opens an account with a bank at 10% rate of compound interest. What amount should he deposit at the end of every month in this recurring account? $[(1.0083)^{240}=6.194]$.

Solution :

Here A = 20,00,000 ; $i=\frac{10}{100}=0.1$ n = 20 and k = 12

$$A = \frac{a}{i/k}\left[\left(1+\frac{i}{k}\right)^{nk}-1\right]$$

$$20,00,000 = \frac{a}{\frac{0.1}{12}}\left[\left(1+\frac{0.1}{12}\right)^{20\times 12}-1\right]$$

$$= \frac{12a}{0.1}\left[\left(1+\frac{0.1}{12}\right)^{240}-1\right]$$

$$= 120a\left[\left(\frac{12.1}{12}\right)^{240}-1\right]$$

$$= 120a\,[(1.0083)^{240}-1]$$

$$= 120a\,[6.194-1]$$

$$= 120a\,(5.194)$$

$$a = \frac{20.00,000}{120\times 5.194}$$

$$= 3,209$$

₹3,209 is to be deposited at the end of every month.

Example 7.4

A person deposits ₹4,000 in the beginning of every year. If the rate of compound interest is 14% then ,find the amount after 10 years . [$(1.14)^{10}$=3.707]

Solution :

Here a = 4,000 ; i = 0.14 and n = 10.

$$A = (1+i)\ \frac{a}{i}\,[(1+i)^{n}-1]$$

$$= (1+0.14)\ \frac{4000}{0.14}\ [(1+0.14)^{10}-1]$$

$$= (1.14)\,\frac{4000}{0.14}\ [(1.14)^{10}-1]$$

$$= 1.14\times\frac{4000}{0.14}\,(3.707-1)$$

$$= 1.14\times\frac{4000}{0.14}\,(2.707) = 88,170.$$

$$A = ₹88,170$$

Example 7.5

A person purchases a machine on 1st January 2009 and agrees to pay 10 installments each of ₹12,000 at the end of every year inclusive of compound rate of 15%. Find the present value of the machine. $[(1.15)^{10}=4.016]$.

Solution :

Here $n=10$, $a = 12{,}000$ and $i = 0.15$

$$\text{Now} \quad P = \frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$= \frac{12{,}000}{0.15}\left[1-\frac{1}{(1+0.15)^{10}}\right]$$

$$= \frac{12{,}000}{0.15}\left[1-\frac{1}{(1.15)^{10}}\right]$$

$$= \frac{12{,}00{,}000}{15}\left[1-\frac{1}{4.016}\right]$$

$$= 80{,}000\left[\frac{4.016-1}{4.016}\right]$$

$$= 80{,}000\left[\frac{3.016}{4.016}\right] = 60{,}080.$$

$$\therefore P = ₹60{,}080$$

Example 7.6

A photographer purchases a camera on installments. He has to pay 7 annual installments each of ₹36,000 right from the date of purchase. If the rate of compound interest is 16% then find the cost price (present value) of the camera. $[(1.16)^{7}=2.2828]$

Solution :

Here $a = 36{,}000$; $n = 7$ and $i = 0.16$

$$\text{Now } P = (1+i)\ \frac{a}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$= (1+0.16)\ \frac{36{,}000}{0.16}\left[1-\frac{1}{(1+0.16)^7}\right]$$

$$= \frac{1.160}{0.16}(36{,}000)\left[1-\frac{1}{(1+0.16)^7}\right]$$

$$= \frac{116\times 36000}{16}\left[1-\frac{1}{2.828}\right]$$

$$= \frac{116\times 36000}{16}\left[\frac{2.828-1}{2.828}\right]$$

$$= \frac{116 \times 36,000 \times 1.828}{16 \times 2.828}$$

$$= \frac{116 \times 36,000 \times 1828}{16 \times 2828} = 1,68,709.$$

$\therefore$ P = ₹1,68,709

Example 7.7

A person has taken a loan of ₹7,00,000 at 16% rate of interest from a finance company. If the repayment period is of 15 years then find the installment he has to pay at the beginning of each month. $[(1.0133)^{180}=9.772]$

Solution :

Here $P = 7,00,000$; $n = 15$; $i = 0.16$ and $k = 12$.

$$P = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left[1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right]$$

$$7,00,000 = \left(1+\frac{0.16}{12}\right)\left(\frac{a}{\frac{0.16}{12}}\right)\left[1-\frac{1}{\left(1+\frac{0.16}{12}\right)^{15\times 12}}\right]$$

$$= \left(\frac{12.16}{12}\right)\left(\frac{12a}{0.16}\right)\left[1-\frac{1}{\left(\frac{12.16}{12}\right)^{180}}\right]$$

$$= \frac{(1216)a}{16}\left[1-\frac{1}{(1.0133)^{180}}\right]$$

$$= \frac{1216}{16}a\left[1-\frac{1}{9.772}\right]$$

$$= \frac{1216}{16}a\left[\frac{9.772-1}{9.772}\right]$$

$$= \frac{1216}{16}a\left[\frac{8.772}{9.772}\right]$$

$$a = \frac{7,00,000 \times 16 \times 9772}{1216 \times 8772} = 10,261$$

Example 7.8

The chairman of a society wishes to award a gold medal to a student getting highest marks in Business Mathematics. If this medal costs ₹9,000 every year and the rate of compound interest is 15% what amount is to be deposited now.

Solution :

Here $a = 9,000$ and $i = 0.15$

$$P = \frac{a}{i}$$

$$= \frac{9000}{0.15}$$

$$= \frac{9,00,000}{15}$$

$$= 60,000.$$

Therefore the amount to be deposited is ₹ 60,000.

Example 7.9

A limited company wants to create a fund to help their employees in critical circumstances. The estimated expenses per month is ₹18,000.Find the amount to be deposited by the company if the rate of compound interest is 15%.

Solution :

Here a = 18,000 ; i = 0.15 and k = 12.

$$P = \frac{\frac{a}{i}}{k}$$

$$= \frac{\frac{18,000}{0.15}}{12}$$

$$= \frac{18,000}{0.15} = \frac{18,00,000 \times 12}{15} = 14,40,000$$

Therefore the amount to be deposited is ₹14,40,000.

Example 7.10

Find the half yearly rate of interest, to get a perpetuity of ₹675 for every half yearly from the present value of ₹30,000/

Solution :

Here P=30,000 ; a = 675 ; k = 2, i = ?

$$P = \frac{\frac{a}{i}}{k}$$

$$30,000 = \frac{675}{\frac{i}{2}}$$

$$= \frac{675}{\frac{i}{2}}$$

$$= \frac{1350}{i}$$

$$= \frac{1350}{30,000}$$

$$= \frac{135}{3000}$$

$$= 0.045$$

Rate of interest $= 0.045 \times 100\%$

$= 4.5\%$

Exercise 7.1

1. Find the amount of an ordinary annuity of ₹3,200 per annum for 12 years at the rate of interest of 10% per year. $[(1.1)^{12} = 3.3184]$

2. If the payment of ₹2,000 is made at the end of every quarter for 10 years at the rate of 8% per year, then find the amount of annuity. $[(1.02)^{40}=2.2080]$

3. Find the amount of an ordinary annuity of 12 monthly payments of ₹1, 500 that earns interest at 12% per annum compounded monthly. $[(1.01)^{12} = 1.1262]$

4. A bank pays 8% per annum interest compounded quarterly. Find the equal deposits to be made at the end of each quarter for 10 years to have ₹30,200 ? $[(1.02)^{40}=2.2080]$

5. A person deposits ₹2,000 from his salary towards his contributory pension scheme. The same amount is credited by his employer also. If 8% rate of compound interest is paid, then find the maturity amount at end of 20 years of service. $[(1.0067)^{240} = 3.3266]$

6. Find the present value of ₹2,000 per annum for 14 years at the rate of interest of 10% per annum. $[(1.1)^{-14}= 0.2632]$

7. Find the present value of an annuity of ₹900 payable at the end of 6 months for 6 years. The money compounded at 8% per annum. $[(1.04)^{-12}=0.6252]$

8. Find the amount at the end of 12 years of an annuity of ₹5,000 payable at the beginning of each year, if the money is compounded at 10% per annum.

9. What is the present value of an annuity due of ₹1,500 for 16 years at 8% per annum?

   $[(1.08)^{15} = 3.1696]$

10. What is the amount of perpetual annuity of ₹50 at 5% compound interest per year?

7.2 Stocks, shares, debentures and Brokerage

To start any big business, a large sum of money is needed. It is generally not possible for an individual to manage such a large sum. Therefore the total sum of money can be divided into equal parts called **shares**. The holder of shares are called **shareholders**.

7.2.1 Types of shares:

There are two type shares namely **common** (or equity) and **preferred**.

The profit gained by the company is distributed among the share holders The preferred share holders have a first claim on dividend. When they have been paid, the remaining profit is distributed among the common share holders.

7.2.2 Definitions

(i) **Capital stock** is the total amount invested to start a company.

(ii) The share purchased by the individual is also called **stock**.

(iii) The persons who buy the shares are also called **stock holders**

(iv) **Face value :** The original value of a share at which the company sells/ buys it to investors is called a face value or nominal value or par value. It is to be noted that the original value of share is printed on the share certificate.

> **DO YOU KNOW?**
> *The market value of a share keeps on changing from time to time.*

(v) **Market value :** The price at which the stock is bought or sold in the market is called the market value (or cash value).

Remarks:

(i) If the market value of a share is greater than the face value then, the share is said to be above par (or at premium).

(ii) If the market value of a share is the same as its face value then, the share is said to be at par.

(iii) If the market value of a share is less than the face value then, the share is said to be below par (or at discount).

Dividend :

The profit gained by a company, distributed among the share holders is called dividend. It is calculated on the face value of the share.

> **DO YOU KNOW?**
> *Dividend is expressed as percentage*

Some useful results:

(i) Investment:

Money invested = number of shares × market value of a share

(ii) Income:

Annual income = number of shares × face value of a share × rate of dividend

(iii) Return percentage (or yield percentage):

$$\text{Percentage of return} = \frac{\textit{Income}}{\textit{Investment}} \times 100$$

(iv) Number of shares:

$$\text{Number of shares purchased} = \frac{\textit{Investment}}{\textit{market value of a share}}$$

Stock exchange:

The place where the shares are traded is called the stock exchange (or) stock market.

Brokerage:

A broker who executes orders to buy and sell shares through a stock market is called Stock Broker. A fee or commission for their service is called the brokerage.

Brokerage is generally based on the face value and expressed as a percentage.

NOTE

(i) When the stock is purchased, brokerage is added to cost price.

(ii) When the stock is sold, brokerage is subtracted from the selling price..

Example 7.11

Find the market value of 325 shares of amount ₹100 at a premium of ₹18.

Solution :

Face value of a share = ₹100

Premium per share = ₹18

$$\text{M.V. of 1 share} = ₹118$$

$$\text{Market value of 325 shares} = \text{number of shares} \times \text{M.V of 1 share}$$

$$= 325 \times 118$$

$$= ₹38,350$$

Therefore market value of 325 shares= ₹38,350.

Example 7.12

A man buys 500 shares of amount ₹100 at ₹14 below par. How much money does he pay?

Solution :

Number of shares	=	500
Face value of a share	=	₹100
Discount	=	₹14
Market value of a share	=	100 – 14 (face value – discount)
	=	₹86.
Market value of 500 shares	=	Number of shares × market value of 1share
	=	500 × 86
	=	43,000
Market value of 500 shares	=	₹43,000

Example 7.13

A person buy 20 shares (par value of ₹10) of a company which pays 9% dividend at such a price that he gets 12% on his money. Find the market value of a share.

Solution :-

Face value of one share	=	₹10
Face value of 20 shares	=	₹200
Dividend	=	$\frac{9}{100} \times 200$

$$= ₹18 \quad \left[S.I = \frac{PNR}{100}, N = 1\right]$$

$$\text{Investment} = \frac{18 \times 100}{1 \times 12} \quad [P = \frac{100I}{NR}, (N=1)$$

$$= ₹150$$

A person purchased 20 shares at ₹150

$$\text{The market value of one share} = ₹\frac{150}{20}$$

$$= ₹7.50$$

Example 7.14

If the dividend received from 10% of ₹25 shares is ₹2000. Find the number of shares.

Solution :

Let x be the number of shares.

Market value of 'x' shares = ₹25 x

$$\text{Now} \quad \frac{10}{100} \times 25x = ₹2{,}000$$

$$\Rightarrow x = \frac{2000 \times 100}{25 \times 10} = 800.$$

Hence the number of shares = 800

Example 7.15

Find the number of shares which will give an annual income of ₹3,600 from 12% stock of face value ₹100.

Solution :

Let 'x' be the number of shares.

Face Value = ₹100

Market value of 'x' shares = ₹100x

$$\frac{12}{100} \times 100x = ₹3600$$

$$12\,x = 3600 \Rightarrow x = 300$$

Hence the number of shares = 300

Example 7.16

A man invest ₹96,000 on ₹100 shares at ₹80. If the company pays him 18% dividend, find

(i) the number of shares he buys

(ii) his total dividend

(iii) his percentage return on the shares.

Solution :-

(i)

$$\text{Investment} = ₹96{,}000$$

$$\text{Face Value} = ₹100$$

$$\text{Market Value} = ₹80$$

$$\text{The number of shares bought} = \frac{Investments}{M.V \text{ } of \text{ } one \text{ } share}$$

$$= \frac{96{,}000}{80}$$

$$= 1200 \text{ shares}$$

(ii)

$$\text{Total dividend} = \text{No. of shares} \times \text{Rate of dividend} \times \text{Face value of one share}$$

$$= 1200 \times \frac{18}{100} \times 100$$

$$= ₹21{,}600$$

(iii)

$$\text{Dividend on ₹ 96000} = ₹21600$$

$$\text{Percentage return on the shares} = \frac{21{,}600}{96{,}000} \times 100$$

$$= \frac{45}{2}$$

$$= 22.5$$

$$\text{Thus return on the shares} = 22.5\%$$

Example 7.17

A person brought at 12% stock for ₹54,000 at a discount of 17%. If he paid 1% brokerage, find the percentage of his income.

Solution :

$$\text{Face value} = ₹100$$

$$\text{Market value} = ₹(100 - 17 + 1)$$

$$= ₹84$$

$$\therefore \quad \text{percentage of his income} = \frac{(12\times100)}{84}$$

$$= \frac{100}{7} = 14\frac{2}{7}$$

$$\therefore \quad \text{income} = 14\frac{2}{7}\ \%$$

Example 7.18

Equal amounts are invested in 10% stock at 89 and 7% stock at 90 (1% brokerage paid in both transactions). If 10% stock bought ₹100 more by way of dividend income than the other, find the amount invested in each stock.

Solution :

Let x be the amount invested in each stock

F.V. = ₹100, M.V. = ₹90

$$\text{Number of shares} = \frac{\textit{investments}}{\textit{M.V.}}$$

$$= \frac{x}{90}$$

$$\text{Annual income} = \frac{x}{90}\times\frac{10}{100}\times 100$$

$$= \frac{x}{9} \qquad \ldots (1)$$

$$\text{M.V.} = ₹91$$

$$\text{Number of shares} = \frac{\text{investments}}{MV}$$

$$= \frac{x}{91}$$

$$\text{Annual income} = \frac{x}{91}\times\frac{7}{100}\times 100$$

$$= \frac{x}{13} \qquad \ldots (2)$$

Given :

$$\frac{x}{9} - \frac{x}{13} = 100$$

$$x = ₹\ 2925$$

$\therefore$ The amount invested in each stock = Rs.2925

Example 7.19

A capital company is made up of 1,00,000 preference shares with a dividend of 16% and 50,000 shares.The par value of each of preference and ordinary shares is ₹10.The total profit of a company is ₹ 3,20,000.If ₹40,000 were kept in reserve and ₹20,000 were kept in depreciation fund, what percent of dividend is paid to the ordinary share holders

Solution:

F.V. = ₹10

Preference shares investments = ₹1,00,000 × 10 = ₹10,00,000

Ordinary shares investments = ₹50,000 × 10 = ₹5,00,000

Total dividend = ₹(3,20,000 – 40,000 – 20,000) = ₹2,60,000

Dividend for preference shares = $\frac{16}{100} \times 10{,}00{,}000$ = ₹1,60,000

Dividend to ordinary shares = 2,60,000 – 1,60,000 = ₹1,00,000

Dividend rate for ordinary share = $\frac{Income}{Investment} \times 100\%$

Dividend = $\frac{1{,}00{,}000}{5{,}00{,}000} \times 100 = 20\%$

Example 7.20

A person sells a 20% stock of face value ₹10,000 at a premium of 42%. With the money obtained he buys a 15% stock at a discount of 22%. What is the change in his income if the brokerage paid is 2%.

Solution:-

F.V. = ₹100

$$\text{Income} = \frac{20}{100} \times 10000$$

$$= ₹\ 2{,}000 \qquad \dots (1)$$

Case (i)

Investment = ₹10,000

$$\text{Face value} = ₹100$$

$$\text{Market value} = ₹100 + 42 - 2 = 140$$

$$\text{Number of shares} = \frac{investments}{FV}$$

$$= \frac{10{,}000}{100}$$

$$= 100$$

$$\text{Sales proceeds} = 100 \times 140$$

$$= 14{,}000$$

Case (ii)

$$\text{M.V.} = ₹100 - 22 + 2$$

$$= 80$$

$$\text{Number of shares} = \frac{investments}{FV}$$

$$= \frac{14000}{80}$$

$$= 175$$

$$\text{Income} = 175 \times \frac{15}{100}$$

$$= ₹2625$$

$$\text{Change of income} = ₹\ 2625 - ₹\ 2000 = ₹\ 625$$

Example 7.21

Which is better investment 12% ₹20 shares at ₹ 16 (or) 15% ₹ 20 shares at ₹24 .

Solution:

Let the investment in each case be ₹ (16× 24)

Case(i)

Income from 12 % ₹20 shares at ₹ 16

$$= \frac{12}{16} \times (16 \times 24)$$

$$= ₹\ 288$$

Case(ii)

Income from 15% ₹20 Shares at ₹ 24

$$= \frac{15}{24} \times (16\times 24)$$

$$= ₹\ 240$$

Hence, the first investment is better.

Exercise 7.2

1. Find the market value of 62 shares available at ₹132 having the par value of ₹100.

2. How much will be required to buy 125 of ₹25 shares at a discount of ₹7

3. If the dividend received from 9% of ₹20 shares is ₹1,620, find the number of shares.

4. Mohan invested ₹29,040 in 15% of ₹100 shares of a company quoted at a premium of 20%. Calculate

   (i) the number of shares bought by Mohan

   (ii) his annual income from shares

   (iii) the percentage return on his investment

5. A man buys 400 of ₹10 shares at a premium of ₹2.50 on each share. If the rate of dividend is 12% find

   (i) his investment

   (ii) annual dividend received by him

   (iii) rate of interest received by him on his money

6. Sundar bought 4,500 of ₹10 shares, paying 2% per annum. He sold them when the price rose to ₹23 and invested the proceeds in ₹25 shares paying 10% per annum at ₹18. Find the change in his income.

7. A man invests ₹13,500 partly in 6% of ₹100 shares at ₹140 and partly in 5% of ₹100 shares at ₹125. If his total income is ₹560, how much has he invested in each?

8. Babu sold some ₹100 shares at 10% discount and invested his sales proceeds in 15% of ₹50 shares at ₹33. Had he sold his shares at 10% premium instead of 10% discount, he would have earned ₹450 more. Find the number of shares sold by him.

9. Which is better investment? 7% of ₹100 shares at ₹120 (or) 8% of ₹100 shares at ₹135.

10. Which is better investment? 20% stock at 140 (or) 10% stock at 70.

Exercise 7.3

IK6S5I

Choose the correct answer

1. The dividend received on 200 shares of face value Rs.100 at 8% dividend value is

   (a) 1600 (b) 1000 (c) 1500 (d) 800

2. What is the amount related is selling 8% stacking 200 shares of face value 100 at 50.

   (a) 16,000 (b) 10,000 (c) 7,000 (d) 9,000

3. A man purchases a stock of ₹ 20,000 of face value 100 at a premium of 20%, then investment is

   (a) ₹ 20,000 (b) ₹ 25,000 (c) ₹ 22,000 (d) ₹ 30,000

4. A man received a total dividend of ₹ 25,000 at 10% dividend rate on a stock of face value Rs.100, then the number of shares purchased.

   (a) 3500 (b) 4500 (c)2500 (d)300

5. The brokerage paid by a person on this sale of 400 shares of face value Rs.100 at 1% brokerage

   (a) ₹ 600 (b) ₹ 500 (c) ₹ 200 (d) ₹ 400

6. Market price of one share of face value 100 available at a discount of $9\frac{1}{2}$ % with brokerage $\frac{1}{2}$ % is

   (a) ₹89 (b) ₹90 (c) ₹91 (d) ₹95

7. A person brought a 9% stock of face value ₹100, for100 shares at a discount of 10%, then the stock purchased is

   (a) ₹9000 (b) ₹6000 (c) ₹5000 (d) ₹4000

8. The Income on 7 % stock at 80 is

(a) 9% (b) 8.75% (c) 8% (d) 7%

9. The annual income on 500 shares of face value 100 at 15% is

(a) ₹7,500 (b) ₹ 5,000 (c) ₹8,000 (d) ₹ 8,500

10. ₹ 5000 is paid as perpetual annuity every year and the rate of C.I 10 %. Then present value P of immediate annuity is

(a) ₹60,000 (b) ₹ 50,000 (c) ₹10,000 (d) ₹ 80,000

11. If 'a' is the annual payment, 'n' is the number of periods and 'i' is compound interest for ₹1 then future amount of the annuity is

(a) A= $\frac{a}{i}(1+i)[(1+i)^n - 1]$ (b) $A = \frac{a}{i}[(1+i)^n - 1]$

(c) $P = \frac{a}{i}$ (d) $P = \frac{a}{i}(1+i)[1-(1+i)^{-n}]$

12. A invested some money in 10% stock at 96. If B wants to invest in an equally good 12% stock, he must purchase a stock worth of

(a) ₹80 (b) ₹115.20 (c) ₹120 (d) ₹125.40

13. An annuity in which payments are made at the beginning of each payment period is called

(a) Annuity due (b) An immediate annuity

(c) perpetual annuity (d) none of these

14. The present value of the perpetual annuity of ₹ 2000 paid monthly at 10 % compound interest is

(a) ₹2,40,000 (b) ₹ 6,00,000 (c) ₹20,40,000 (d) ₹ 2,00,400

15. Example of contingent annuity is

(a) Life insurance premium

(b) An endowment fund to give scholarships to a student

(c) Personal loan from a bank

(d) All the above

Miscellaneous Problems

1. Find the amount of annuity of ₹2000 payable at the end of each year for 4 years of money is worth 10% compounded annually [log(1.1) = 0.0414 ; antilog(0.1656) = 1.464]

2. An equipment is purchased on an installment basis such that ₹5000 on the signing of the contract and four yearly installments of ₹300 each payable at the end of first, second, third and the fourth year. If the interest is charged at 5% p.a find the cash down price. [log(1.05) = 0.0212 ; antilog($\bar{1}$.9152=0.8226)]

3. (i) Find the amount of an ordinary annuity of ₹500 payable at the end of each year for 7 years at 7% per year compounded annually.

   (ii) Calculate the amount of an ordinary annuity of ₹10,000 per annum for 5 years at 10% per year compounded half-yearly.

   (iii) Find the amount of an ordinary annuity of ₹600 is made at the end of every quarter for 10 years at the rate of 4% per year compounded quarterly.

   (iv) Find the amount of an annuity of ₹2000 payable at the end of every month for 5 years if money is worth 6% per annum compounded monthly.

4. Naveen deposits ₹250 at the beginning of each month in an account that pays an interest of 6% per annum compounded monthly, how many months will be required for the deposit to amount to atleast ₹6390?

5. A cash prize of ₹1,500 is given to the student standing first in examination of Business Mathematics by a person every year. Find out the sum that the person has to deposit to meet this expense. Rate of interest is 12% p.a

6. Machine A costs ₹15,000 and machine B costs ₹20,000. The annual income from A and B are ₹4,000 and ₹7,000 respectively. Machine A has a life of 4 years and B has a life of 7 years. Find which machine may be purchased. (Assume discount rate 8% p.a)

7. Vijay wants to invest ₹27,000 in buying shares. The shares of the following companies are available to him. ₹100 shares of company A at par value ; ₹100 shares of company B at a premium of ₹25. ₹100 shares of company C at a discount of ₹10. ₹50 shares of company D at a premium of 20%. Find how many shares will he get if he buys shares of company (i) A (ii) B (iii) C (iv) D

8. Gopal invested ₹8,000 in 7% of ₹100 shares at ₹80. After a year he sold these shares at ₹75 each and invested the proceeds (including his dividend) in 18% for ₹25 shares at ₹41. Find

   (i) his dividend for the first year

   (ii) his annual income in the second year

   (iii) The percentage increase in his return on his original investment

9. A man sells 2000 ordinary shares (par value ₹10) of a tea company which pays a dividend of 25% at ₹33 per share. He invests the proceeds in cotton textiles (par value ₹25) ordinary shares at ₹44 per share which pays a dividend of 15%. Find (i) the number of cotton textiles shares purchased and (ii) change in his dividend income.

10. The capital of a company is made up of 50,000 preferences shares with a dividend of 16% and 2,500 ordinary shares. The par value of each of preference and ordinary shares is ₹10. The company had a total profit of ₹1,60,000. If ₹20,000 were kept in reserve and ₹10,000 in depreciation, what percent of dividend is paid to the ordinary share holders

Summary

Types of Annuities

- Certain Annuity
  - Annuity Due (Payments are made at the beginning of each payment period)
    $$A = \frac{a}{i}(1+i)\left[(1+i)^n - 1\right]$$
    - Present worth
      $$P = \frac{a}{i}(1+i)\left[1-(1+i)^{-n}\right]$$
- Contingent Annuity
  - Immediate Annuity (or) Ordinary Annuity (Payment are made at the end of each payment period)
    $$A = \frac{a}{i}\left[(1+i)^n - 1\right]$$
    - Present worth
      $$P = \frac{a}{i}\left[1-(1+i)^{-n}\right]$$
- Perpetual Annuity
  - Present worth
    $$P = \frac{a}{i}$$

- **Endowment or Scholarship fund**

  (i) If the scholarship is awarded endlessly, then $P = \frac{a}{i}$

  (ii) If the scholarship is awarded for a fixed period, say, n years, then

  $$P = \frac{a}{i}\left[1-\left(1+i\right)^{-n}\right]$$

- **Face Value:** The original Value of the share is called its nominal values or face value or printed value.

- **Market Value:** The price at which the share is sold (or) purchased in the capital market through stock exchanges is called the market value.

- A Share is called at par if the market value of the share is equal to its face (or) nominal value.

- A share is said to be above par (or) at premium, if the market value of the share is more than its nominal value.

- A share is said to be below par (or) at discount, if the market value of the share is less than its nominal value.

- The part of the annual profit, which a share holder gets for his investment from the company is called dividend.

- Dividend is always declared on the face value of the share and the rate of dividend is expressed as a percentage of the nominal value of a share per annum.

- Annual income of a shareholders = $\frac{n \times r \times F.V}{100}$

  Where n = number of shares with the shareholders

  r = rate of dividend,

- Annual Return = $\frac{\text{Annual income}}{\text{investment in shares}} \times 100\%$

- Number of Shares held = $\frac{\text{investment}}{M.V(or)F.V. of \text{ one share(as the type of investment)}}$

(or)

$$= \frac{\text{Annual income}}{\text{income from one share}} \text{ (or) } = \frac{\text{Total } F.V}{F.V. of \text{ one share}}$$

- Which is better investment for two stocks

Let the investment each case be = (M.V of First stock × M.V. of Second Stock)

Case (i)

Income form r_1 % = $\frac{r_1}{(investment\ in\ each\ stock)}$ × Total investment

Case (ii)

Income form r_2 % = $\frac{r_2}{(investment\ in\ each\ stock)}$ ×Total investment

GLOSSARY	
Brokerage	தரகு
Capital value	மூலதன மதிப்பு
Debentures	கடன் பத்திரங்கள்
Equity shares	சம பங்கு
Face value	முக மதிப்பு
Immediate annuity	தவணை பங்கீட்டு தொகை
Interest	வட்டி
Market price	சந்தை விலை
Payment interval	செலுத்தும் கால இடைவெளி
Periodic payment	காலமுறை செலுத்துதல்
Perpetual annuity	நிரந்தர தவணை பங்கீட்டு தொகை
Preference shares	முன்னுரிமை பங்குகள்
Selling price	விற்பனை விலை
Share holders	பங்குதாரர்கள்
Shares	பங்குகள்
Stock exchange	பங்குச் சந்தை
Stocks	சரக்கு முதல்கள்
Term of annuity	தவணை பங்கீட்டு தொகை காலம்
Transaction	பரிவர்த்தனை

ICT Corner

Financial Mathematics

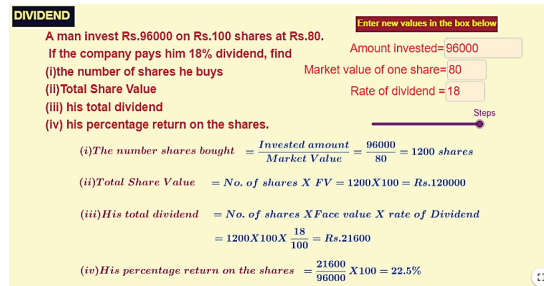
Step – 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "11th Business Maths Volume-2" will appear. In that there are several worksheets related to your Text Book.

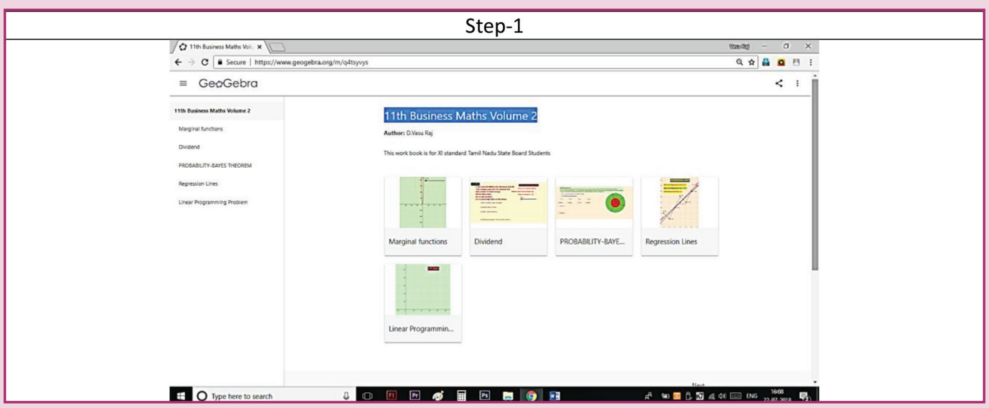
Step-2

Select the work sheet "Dividend" Move the sliders to see the steps. Type the Amount invested, Market value of one share and dividend rate in the box seen on right side and proceed.

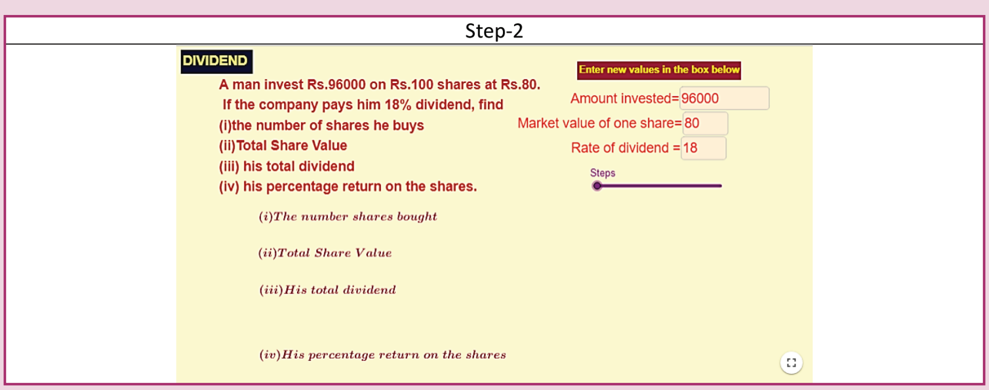


Expected Outcome ⇒

Step 1



Step 2



Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

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Chapter 8

Descriptive statistics and probability

Learning Objectives

After studying this chapter students will be able to understand the following concepts.

- Measures of Central Tendency like A.M, G.M & H.M
- Relationship among the averages.
- Related positional measures like Quartiles, Deciles and percentiles etc.,
- Measures of Dispersion like Quartile Deviation, mean Deviation
- Relative measures like co-efficient of Quartile Deviation, co-efficient of Mean Deviation
- Concept of conditional probability and multiplication theorem.
- Baye's theorem and its applications.

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8.1 Measures of central tendency

Introduction:

One of the most important objectives of Statistical analysis is to get one single value that describes the characteristic of the entire value for data. Such a value represent the measure of central tendency for the complete data. The word average is very commonly used in day-to-day conversation. For example, we often talk of average boy in a class, average height or average life of an Indian, average income, etc. Sir Ronald Fisher who is known to be a father of statistics and he made his pioneering contributions in the applications of statistics in various disciplines.

8.1.1 Average

- Recall

There are several measures of central tendency for the data. They are

- Arithmetic Mean
- Median
- Mode
- Geometric Mean
- Harmonic Mean

Arithmetic Mean (discrete case)

Arithmetic mean of a set of observations is their sum divided by the number of observations. The observation are classified into a) Ungrouped data and b) Grouped data.

a) Ungrouped data

(i) Direct Method:

$$\overline{X} = \frac{X_1 + X_2 + X_3 + \ldots X_n}{n} = \frac{\sum X}{n}$$

where $\overline{X}$ is Arithmetic Mean, $\sum X$ is sum of all the values of the variable X and n is number of observations.

(ii) Short-cut method

The arithmetic mean can be calculated by using any arbitrary value A as origin and write d as the deviation of the variable X then,

$\overline{X} = A + \frac{\sum d}{n}$ where $d = X - A$.

b) Grouped data

(i) Direct method

The formula for computing the mean is $\overline{X} = \frac{\sum fx}{N}$

where f is frequency; X is the variable; $N = \sum f$ i.e. total frequency.

(ii) Short-cut method

The arithmetic mean is computed by applying the following formula:

$\overline{X} = A + \frac{\Sigma fd}{N}$; where A is assumed mean; $d = X - A$; $N = \sum f$

Arithmetic mean for Continuous case

The arithmetic mean may be computed by applying any of the following methods:

(i) Direct method

(ii) Short-cut method

(iii) Step deviation method

(i) Direct method

When direct method is used arithmetic mean is defined as

$$\overline{X} = \frac{\Sigma fm}{N}$$

Where m = midpoint of each of the class interval,

f = the frequency of each class interval

$N = \sum f =$ total frequency

(ii) Short-cut method

The arithmetic mean is computed by applying the following formula

$$\overline{X} = A + \frac{\Sigma fd}{N}$$

where A is assumed mean or arbitrary value, $d = m - A$ is deviations of mid-point from assumed mean and $N = \sum f$

(iii) Step Deviation Method

In case of grouped or continuous frequency distribution, the arithmetic mean

$\overline{X} = A + \left(\frac{\Sigma fd}{N} \times c\right)$, where $d = \frac{(m - A)}{c}$ A is any arbitrary value or assumed mean and c is the magnitude of class interval.

DO YOU KNOW?

All the above three methods of finding arithmetic mean in continuous case gives us the same answer.

Mode:

Mode is the value which repeats maximum number of times among the given observations.

Median:

Median is exactly a middle value and it exceeds and exceeded by the same number of observations. Median is one of the positional measure. Some other related positional measures are also described below.

NOTE

It is believed that the students might be familiar with the above concepts and our present syllabus continues from the following.

8.1.2 Related Positional Measures - Quartiles, Deciles and Percentiles :

Besides median there are other measures which divide a series into equal parts. Important amongest these are quartiles, deciles and percentiles.

(i) Quartiles:

A measure which divides an array into four equal parts is known as quartiles.

Each portion contains equal number of items. The first, second and third points are termed as first quartile (Q_1), second quartile (Q_2) (better named as median) and the third quartile(Q_3). The first quartile (Q_1) or lower quartile, has 25% of the items of the distribution below it and 75% of the items are greater than it. Q_2(median) the second quartile or median has 50 % of the observations above it and 50% of the observations below it. The upper quartile or third quartile(Q_3) has 75% of the items of the distribution below it and 25% of the items are above it. Similarly the other two positional measures can be defined.

(ii) Deciles:

A measure which divides an array into ten equal parts is known as deciles.

That is Deciles are the values which divides the series into ten equal parts. We get nine dividing positions namely $D_1,D_2,\ldots,D_9$ which are called as deciles. Therefore there are nine deciles. It is to be noted that D_5 is equal to median.

(iii) Percentiles:

A measure which divides an array into hundred equal parts is known as Percentiles.

That is percentiles are the values which divides the series into hundred equal parts. We get 99 dividing positions $P_1, P_2, \ldots, P_{99}$ which are called as percentiles. Therefore there are 99 percentiles. It is to be noted that P_{50} is equal to median.

8.1.3 Computations for Related positional measure

The procedure for computing quartiles, deciles and percentiles are the same as the median.

(i) Ungrouped data:

Steps:

1. Arrange the data either in ascending or descending order of magnitude.

2. Apply the formula

$$Q_1 = \text{Size of } \left(\frac{n+1}{4}\right)^{th} \text{ value}$$

$$Q_3 = \text{Size of } \left(\frac{3(n+1)}{4}\right)^{th} \text{ value}$$

$$D_1 = \text{Size of } \left(\frac{n+1}{10}\right)^{th} \text{ value}$$

$$D_2 = \text{Size of } 2\left(\frac{n+1}{10}\right)^{th} \text{ value}$$

$$P_{60} = \text{Size of } 60\left(\frac{n+1}{100}\right)^{th} \text{ value}$$

$$P_{99} = \text{Size of } 99\left(\frac{n+1}{100}\right)^{th} \text{ value}$$

DO YOU KNOW?

$Q_2 = D_5 = P_{50} =$ **Median**

Example 8.1

Find D_2 and D_6 for the following series 22, 4, 2, 12, 16, 6, 10, 18, 14, 20, 8

Solution :

Here $n = 11$ observations are arranged into ascending order

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

$$D_2 = \text{size of } 2\left(\frac{n+1}{10}\right)^{th} \text{ value}$$

$$D_6 = \text{size of } 6\left(\frac{n+1}{10}\right)^{th} \text{ value}$$

$D_2 =$ size of 2.4th value $\approx$ size of 2nd value $= 4$

$D_6 =$ size of 7.2^{th} value $\approx$ size of 7^{th} value $= 14$

Example 8.2

Calculate the value of Q_1, Q_3, D_6 and P_{50} from the following data

Roll No	1	2	3	4	5	6	7
Marks	20	28	40	12	30	15	50

Solution:

Marks are arranged in ascending order

12 15 20 28 30 40 50

n = number of observations =7

Q_1= Size of $\left(\frac{n+1}{4}\right)^{th}$ value = Size of $\left(\frac{7+1}{4}\right)^{th}$ value

= Size of 2nd value = 15

Q_3 = Size of $\left(\frac{3(n+1)}{4}\right)^{th}$ value

= Size of$\left(\frac{3\times 8}{4}\right)^{th}$ value

= Size of 6th value = 40

D_6 = Size of $\left(\frac{6(n+1)}{10}\right)^{th}$ value

= Size of $\left(\frac{6\times 8}{10}\right)^{th}$ value = Size of 4.8th value Size of 5th value = 30

P_{50} = Size of $\left(\frac{50(n+1)}{100}\right)^{th}$ value = Size of 4th value = 28

Hence Q_1=15, Q_3=40, D_6 =30 and P_{50}=28

(ii) Grouped data (discrete case):

Steps:

1. Arrange the data in ascending or descending order of magnitude.

2. Find out cumulative frequencies.

3. Apply the formula:

Q_1 = Size of $\left(\frac{N+1}{4}\right)^{th}$ value

Q_3 = Size of $\left(\frac{3(N+1)}{4}\right)^{th}$ value

Now look at the cumulative frequency (*cf*) column and find that total which is either equal to $\left(\frac{N+1}{4}\right)$ or next higher than that and determine the value of the variable corresponding to this. That gives the value of Q_1. Similarly Q_3 is determined with reference of $\frac{3(N+1)}{4}$ value of the variable.

Example 8.3

Compute Q_1, D_2 and P_{90} from the following data

Marks	10	20	30	40	50	60
No. of Students	4	7	15	8	7	2

Solution :

Marks X	Frequency f	Cumulative Frequency cf
10	4	4
20	7	11
30	15	26
40	8	34
50	7	41
60	2	N= 43

Table : 8.1

$$Q_1 = \text{Size of } \left(\frac{N+1}{4}\right)^{th} \text{ value} = \text{Size of } \frac{43+1}{4} = 11^{\text{th}} \text{ value} = 20$$

$$D_2 = \text{Size of } \left(\frac{2(N+1)}{10}\right)^{th} \text{ value} = \text{Size of } \frac{88}{10} = 8.8^{\text{th}} \text{ value} = 20$$

$$P_{90} = \text{Size of } \left(\frac{90(N+1)}{100}\right)^{th} \text{ value} = \text{Size of } \frac{3960}{100} = 39.6^{\text{th}} \text{ value} = 50$$

(iii) Grouped data (Continuous case):

In the case of continuous frequency distribution, the classes are arranged either in ascending or descending order and the class corresponding to the cumulative frequency (cf) just equal or greater than (N/4) is called Q_1 class and the value of Q_1 is obtained by the following formula:

$$Q_1 = L + \left(\frac{\frac{N}{4} - pcf}{f}\right) \times c$$

where L is the lower limit of the Q_1 class,

f is the frequency of the Q_1 class,

c is the magnitude of the Q_1 class,

pcf is the cumulative frequency of the pre Q_1 class.

Similarly third quartile value can be obtained by the same procedure with Q_3class by the following formula:

$$Q_3 = L + \left(\frac{\frac{3N}{4} - pcf}{f} \right) \times c$$

where L is the lower limit of the third quartile class,

f is the frequency of the third quartile class,

c is the magnitude of the third quartile class,

pcf is the cumulative frequency of the pre Q_3 class

Similarly the same procedure is to be followed for other positional measures such as deciles and percentiles,

$$D_4 = L + \left(\frac{\frac{4N}{10} - pcf}{f} \right) \times c$$

$$P_{60} = L + \left(\frac{\frac{60N}{100} - pcf}{f} \right) \times c$$

Example 8.4

Compute upper Quartiles, lower Quartiles, D_4 and P_{60}, P_{75} from the following data

CI	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Frequency	12	19	5	10	9	6	6

Solution:

CI	*f*	*cf*
10 – 20	12	12
20 – 30	19	31
30 – 40	5	36
40 – 50	10	46
50 – 60	9	55
60 – 70	6	61
70 – 80	6	N= 67
	N = 67	

Table : 8.2

$Q_1 =$ Size of $\left(\frac{N}{4}\right)^{th}$ value $= \frac{67}{4} =$ 16.75th value.

Thus Q_1 lies in the class (20 – 30) and its corresponding values are $L = 20$;

$\frac{N}{4} = 16.75$; $pcf = 12$; $f = 19$; $c = 10$

$$Q_1 = L + \left(\frac{\frac{N}{4} - pcf}{f}\right) \times c$$

$$Q_1 = 20 + \left(\frac{16.75 - 12}{19}\right) \times 10 = 20 + 2.5 = 22.5$$

$Q_3 =$ Size of $\left(\frac{3N}{4}\right)$th value = 50.25th value

So Q_3 lies in the class (50-60) corresponding values are $L = 50$, $\left(\frac{3N}{4}\right) = 50.25$; $pcf = 46$, $f = 9$, $c = 10$

$$Q_3 = L + \left(\frac{\frac{3N}{4} - pcf}{f}\right) \times c$$

$$Q_3 = 50 + \left(\frac{50.25 - 25}{9}\right) \times 10 = 54.72$$

$$D_4 = L + \left(\frac{\frac{4N}{10} - pcf}{f}\right) \times c$$

$D_4 =$ Size of $\left(\frac{4N}{10}\right)$th value = 26.8th value.Thus D_4 lies in the class (20 – 30) and its corresponding values are $L = 20$, $\frac{4N}{10} = 26.8$; $pcf = 12$, $f = 19, c = 10$.

$$D_4 = 20 + \left(\frac{26.8 - 12}{19}\right) \times 10$$

$$= 27.79$$

$P_{75} =$ Size of $\left(\frac{75N}{100}\right)$th value = 50.25th value.Thus P_{75} lies in the class (50 – 60) and its corresponding values are $L = 50$; $\frac{75N}{100} = 50.25$; $pcf = 46$, $f = 9, c = 10$.

$$P_{75} = L + \left(\frac{\frac{75N}{100} - pcf}{f}\right) \times c$$

$$= 50 + \left(\frac{50.25 - 46}{9}\right) \times 10$$

$$= 54.72$$

8.1.4 Geometric mean

Geometric mean is defined as the n^{th} root of the product of *n* observations or values.

If there are two observations, we take the square root; if there are three observations we have to take the cube root and so on

$$GM = \sqrt[n]{X_1 . X_2 . X_3 .....X_n} = (X_1, X_2, X3, ...X_n)^{1/n}$$

where $X_1, X_2, X_3, ..., X_n$ refer to the various items of the series which are all greater than zero and n refers number of observations.

Thus the geometric mean of 3 values 2,3,4 would be

$$G\,M = \sqrt[3]{(2)(3)(4)} \quad = 2.885$$

When the number of items is three or more the task of multiplying the numbers and of extracting the root becomes excessively difficult. To simplify calculations, logarithms are used. Geometric mean is calculated as follows:

$$\log GM = \frac{\log X_1 + \log X_2 + ... + \log X_n}{n}$$

$$\text{(or)} \quad \log GM = \left(\frac{\Sigma \log X}{n}\right)$$

$$GM = Anti\log\left(\frac{\Sigma \log X}{n}\right)$$, n is number of observation.

(i) In discrete observation

$$GM = Anti\log\left(\frac{\Sigma f \log X}{N}\right); \text{ where } N = \Sigma f$$

(ii) In Continuous observation

$$\text{GM} = Anti\log\left[\frac{\Sigma f \log m}{N}\right]; \text{ where } m \text{ is midpoint and } N = \Sigma f$$

Example 8.5

Daily income (in Rs) of ten families of a particular place is given below. Find out GM

85, 70, 15, 75, 500, 8, 45, 250, 40, 36

Solution:

X	log X
85	1.9294
70	1.8451
15	1.1761
75	1.8751
500	2.6990
8	0.9031
45	1.6532
250	2.3979
40	1.6021
36	1.5563
	$\Sigma \log X = 17.6373$

Table : 8.3

$$GM = Anti\log\left(\frac{\Sigma \log X}{n}\right);\text{ where } n = 10$$

$$GM = Anti\log\left(\frac{17.6373}{10}\right)$$

$$= Anti\log(1.7637)$$

$$GM = 58.03$$

Example 8.6

Calculate the geometric mean of the data given below giving the number of families and the income per head of different classes of people in a village of Kancheepuram District.

Class of people	No. of Families	Income per head in 1990 (Rs)
Landlords	1	1000
Cultivators	50	80
Landless labourers	25	40
Money- lenders	2	750
School teachers	3	100
Shop-keepers	4	150
Carpenters	3	120
Weavers	5	60

Solution:

Calculation of Geometric Mean

Class of people	Income per head in 1990(Rs) X	No of Families f	log X	f log X
Landlords	1000	1	3.0000	3.0000
Cultivators	80	50	1.9031	95.1550
Landless labourers	40	25	1.6021	40.0525
Money- lenders	750	2	2.8751	5.7502
School teachers	100	3	2.0000	6.0000
Shopkeepers	150	4	2.1761	8.7044
Carpenters	120	3	2.0792	6.2376
Weavers	60	5	1.7782	8.8910
		N = 93		$\Sigma f \log X = 173.7907$

Table : 8.4

$$G\,M = Anti\log\left(\frac{\Sigma f \log X}{N}\right)$$

$$= Anti\log\left(\frac{173.7907}{93}\right)$$

$$= Anti\log(1.8687)$$

GM = 73.95

Example 8.7

Compute the Geometric mean from the data given below

Marks	0-10	10-20	20-30	30-40	40-50
No. of Students	8	12	18	8	6

Solution:

Marks	m	f	log m	f log m
0-10	5	8	0.6990	5.5920
10-20	15	12	1.1761	14.1132
20-30	25	18	1.3979	25.1622
30-40	35	8	1.5441	12.3528
40-50	45	6	1.6532	9.9192
		N = 52		$\Sigma f \log m = 67.1394$

Table : 8.5

$$GM = Anti\log\left(\frac{\Sigma f \log m}{N}\right)$$

$$= Anti\log\left(\frac{67.1394}{52}\right)$$

$$= Anti\log(1.2911)$$

$$GM = 19.55$$

Specific uses of Geometric mean

The most useful application of geometric mean is to average the rate of changes. For example, from 2006 to 2008 prices increased by 5%, 10% and 18% respectively. The average annual increase is not 11% $\left(\frac{5+10+18}{3}=11\right)$ as given by the arithmetic average but 10.9% as obtained by the geometric mean. This average is also useful in measuring the growth of population, because population increases in geometric progression.

Example 8.8

Compared to the previous year the overhead expenses went up by 32% in 1995, they increased by 40% in the next year and by 50% in the following year. Calculate the average rate of increase in overhead expenses over the three years.

Solution:

In averaging ratios and percentages, geometric mean is more appropriate. Let us consider X represents Expenses at the end of the year.

% Rise	X	$\log X$
32	132	2.1206
40	140	2.1461
50	150	2.1761
		$\Sigma \log X = 6.4428$

Table : 8.6

$$GM = Anti\log\left(\frac{\Sigma \log X}{n}\right)$$

$$= Anti\log\left(\frac{6.4428}{3}\right)$$

$$= Anti\log(2.1476)$$

$GM = 140.5$

Average rate of increase in overhead expenses

140.5 – 100 = 40.5 %

DO YOU KNOW?

Geometric mean cannot be calculated if one of the observations is zero.

8.1.5 Harmonic mean

Harmonic mean is defined as the reciprocal of the arithmetic mean of the reciprocal of the individual observations. It is denoted by HM

Thus, HM = $$\frac{n}{\left(\frac{1}{X_1}+\frac{1}{X_2}+.....+\frac{1}{X_n}\right)}$$

When the number of items is large the computation of harmonic mean in the above manner becomes tedious. To simplify calculations we obtain reciprocals of the various items from the tables and apply the following formulae:

(i) In individual observations

$$HM = \frac{n}{\left(\frac{1}{X_1}+\frac{1}{X_2}+.....+\frac{1}{X_n}\right)} \quad \text{Or} \quad HM = \frac{n}{\Sigma\left(\frac{1}{X}\right)}$$

where n is number of observations or items or values

(ii) In discrete frequency distribution

$$HM = \frac{N}{\Sigma\left(\frac{f}{X}\right)}$$ where N = total frequency= $\sum f$

(iii) In continuous frequency distribution

$$HM = \frac{N}{\Sigma\left(\frac{f}{m}\right)}$$

Where m is midpoint and N is total frequency

Example 8.9

Calculate the Harmonic Mean of the following values:

1, 0.5, 10, 45.0, 175.0, 0.01, 4.0, 11.2

Solution:

X	$\frac{1}{X}$
1	1.0000
0.5	2.0000
10	0.1000
45	0.0222
175	0.0057
0.01	100.0000
4.0	0.2500
11.2	0.0893
	$\Sigma\left(\frac{1}{X}\right)=103.4672$

Table : 8.7

$$n = 8$$

$$HM = \frac{n}{\Sigma\left(\frac{1}{X}\right)} = \frac{8}{103.467} = 0.077$$

Example 8.10

From the following data compute the value of Harmonic Mean .

Marks	10	20	25	40	50
No. of students	20	30	50	15	5

Solution:

Calculation of Harmonic Mean

Marks X	No. of Students f	$\frac{f}{x}$
10	20	2.000
20	30	1.500
25	50	2.000
40	15	0.375
50	5	0.100
	$N = 120$	$\Sigma\left(\frac{f}{X}\right) = 5.975$

Table : 8.8

$$HM = \frac{N}{\Sigma\left(\frac{f}{X}\right)} \qquad = \frac{120}{5.975} = 20.08$$

Example 8.11

Calculate Harmonic Mean for the following data given below

Value	0-10	10-20	20-30	30-40	40-50
Frequency	8	12	20	6	4

Solution:

Calculation of Harmonic Mean

Value	Mid-Point m	f	$\left(\frac{f}{m}\right)$
0-10	5	8	1.60
10-20	15	12	0.80
20-30	25	20	0.80
30-40	35	6	0.17
40-50	45	4	0.09
		N = 50	$\Sigma\left(\frac{f}{m}\right)$**=3.46**

Table : 8.9

$$HM = \frac{N}{\Sigma\left(\frac{f}{m}\right)} \qquad = \frac{50}{3.46} = 14.45$$

Special applications of Harmonic Mean

The Harmonic Mean is restricted in its field of usefulness. It is useful for computing the average rate of increase of profits of a concern or average speed at which a journey has been performed or the average price at which an article has been sold. The rate usually indicates the relation between two different types of measuring units that can be expressed reciprocally.

For example, if a man walked 20km in 5 hours, the rate of his walking speed can be expressed

$$\frac{20\,km}{5\,hours} = 4\,km \text{ per hour}$$

where the units of the first term is a km and the unit of the second term is an hour or reciprocally,

$$\frac{5\,hours}{20\,km} = \frac{1}{4} \text{ hour per km}.$$

where the unit of the first term is an hour and the unit of the second term is a km

Example 8.12

An automobile driver travels from plain to hill station 100km distance at an average speed of 30km per hour. He then makes the return trip at average speed of 20km per hour what is his average speed over the entire distance (200km)?

Solution:

If the problem is given to a layman he is most likely to compute the arithmetic mean of two speeds

$$\text{i.e., } \overline{X} = \frac{30\,km + 20\,km}{2} = 25 \;kmph$$

But this is not the correct average.

So harmonic mean would be mean suitable in this situation. Harmonic Mean of 30 and 20 is

$$HM = \frac{2}{\left(\frac{1}{20}\right)+\left(\frac{1}{30}\right)} = \frac{2}{\left(\frac{5}{60}\right)} = \frac{2(60)}{5}$$

$$= 24 \;kmph$$

NOTE

The Harmonic Mean is a measure of central tendency for data expressed as rates such as kilometres per hour, kilometres per litre, periods per semester, tonnes per month etc.,

8.1.6 Relationship among the averages

In any distribution when the original items differ in size, the values of AM, GM and HM would also differ and will be in the following order

$$AM \geq GM \geq HM$$

DO YOU KNOW?

If all the numbers X_1, X_2, …, X_n are identical then, AM = GM = HM.

Example 8.13

Verify the relationship among AM, GM and HM for the following data

X	7	10	13	16	19	22	25	28
f	10	22	24	28	19	9	12	16

Solution:

X	f	Xf	$\log X$	$f\log X$	f/X
7	10	70	0.8451	8.4510	1.4286
10	22	220	1	22.0000	2.2000
13	24	312	1.1139	26.7346	1.8462
16	28	448	1.2041	33.7154	1.7500
19	19	361	1.2788	24.2963	1.0000
22	9	198	1.3424	12.0818	0.4091
25	12	300	1.3979	16.7753	0.4800
28	16	448	1.4472	23.1545	0.5714
	$\sum f = N = 140$	$\sum fX = 2357$		$\sum f\log x = 167.209$	$\sum \frac{f}{x} = 9.6852$

Table : 8.10

$$AM = \frac{\sum fX}{N} = \frac{2357}{140} = 16.84$$

$$GM = Anti\log\left(\frac{\sum f\log X}{N}\right) = Anti\log\left(\frac{167.209}{140}\right) = Anti\log(1.1944) = 15.65$$

$$HM = \frac{N}{\Sigma\left(\frac{f}{X}\right)} = \frac{140}{9.6852} = 14.46$$

i.e. $16.84 > 15.65 > 14.46$

$\therefore\ AM > GM > HM$

Example 8.14

A's scooter gives an average of 40km a litre while B's scooter gives an average of 30km a litre . Find out the mean, if

(i) each one of them travels 120 km.

(ii) the petrol consumed by both of them is 2 litres per head.

Solution:

(i) Here the distance is constant. Hence harmonic mean is appropriate.

$$HM = \frac{n}{\frac{1}{a}+\frac{1}{b}}$$

$$= \frac{2}{\frac{1}{40}+\frac{1}{30}} = \frac{2}{\frac{7}{120}}$$

$$= \frac{2\times 120}{7} = 34.3 \text{ km per litre}$$

(ii) Here the quantity of petrol consumed is fixed i.e 2 litres. Here the arithmetic mean will give the correct answer.

$$\overline{X} = \frac{Total\ distance\ Covered}{Total\ Petrol\ Consumed}$$

$$\overline{X} = \frac{40\times 2+30\times 2}{4} = 35$$

$\therefore$ Average speed = 35km per litre.

Example 8.15

A person purchases tomatoes from each of the 4 places at the rate of 1kg., 2kg., 3kg., and 4kg. per rupee respectively .On the average, how many kilograms has he purchased per rupee?

Solution:

Since we are given rate per rupee, harmonic mean will give the correct answer.

$$HM = \frac{n}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}$$

$$= \frac{4}{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}$$

$$= \frac{4\times 12}{25}$$

$= 1.92$ kg per rupee.

8.2 Measures of dispersion

Average gives us an idea on the point of the concentration of the observations about the central part of the distribution. If we know the average alone we cannot form a complete idea about the distribution as will be clear from the following example.

Consider the series (i) 7,8,9,10,11 (ii) 3,6,9,12,15 and (iii) 1,5,9,13,17. In all these cases we see that n, the number of observations is 5 and the mean is 9. If we are given that the mean of 5 observations is 9, we cannot form an idea as to whether it is the average of first series or second series or third series or of any other series of 5 observations whose sum is 45. Thus we see that the measures of central tendency are inadequate to give us a complete idea of the distribution. So they must be supported and supplemented by some other measures. One such measure is Dispersion, which provides the nature of spreadness of the data.

Literal meaning of dispersion is “scatteredness” we study dispersion to have an idea about the homogeneity or heterogeneity of the distribution. In the above case we say that series (i) is more homogeneous (less dispersed) than the series (ii) or (iii) or we say that series (iii) is more heterogeneous (more scattered) than the series (i) or (ii).

Various measures of dispersion can be classified into two broad categories.

(a) The measures which express the spread of observations in terms of distance between the values of selected observations. These are also termed as distance measures.

Example: Range and interquartile range (or) quartile deviation.

(b) The measures which express the spread of observations in terms of the average of deviations of observation from some central value,

Example: Mean deviation and Standard deviation.

8.2.1 Quartile Deviation

Quartile Deviation is defined as $QD = \frac{1}{2}(Q_3 - Q_1)$. It may also be called as semi-inter quartile.

where Q_1 and Q_3 are the first and third quartiles of the distribution respectively and $Q_3 - Q_1$ is called as inter quartile range.

(i) Relative measures for QD

Quartile deviation is an absolute measure of dispersion. The relative measure corresponding to this measure, called the coefficient of quartile deviation is calculated as follows:

$$\text{Coefficient of QD} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

Coefficient of quartile deviation can be used to compare the degree of variation in different distributions.

(ii) Computation of Quartile Deviation

The process of computing quartile deviation is very simple since we just have to compute the values of the upper and lower quartiles that is Q_3 and Q_1 respectively.

Example 8.16

Calculate the value of quartile deviation and its coefficient from the following data

Roll No.	1	2	3	4	5	6	7
Marks	20	28	40	12	30	15	50

Solution :

Marks are arranged in ascending order

12 15 20 28 30 40 50

n = number of observations =7

Q_1= Size of $\left(\frac{(n+1)}{4}\right)^{th}$ value = Size of $\left(\frac{7+1}{4}\right)^{th}$ value = Size of 2^{nd} value = 15

Hence $Q_1 = 15$

Q_3 = Size of $\left(\frac{3(n+1)}{4}\right)^{th}$ value = Size of $\left(\frac{3 \times 8}{4}\right)^{th}$ value = Size of 6^{th} value = 40

Hence $Q_3 = 40$

$$QD = \frac{1}{2}(Q_3 - Q_1) = \frac{40-15}{2} = 12.5$$

$$\text{Coefficient of } QD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{40-15}{40+15} = \frac{25}{55} = 0.455$$

Hence coefficient of QD = 0.455

Example 8.17

Compute coefficient of quartile deviation from the following data

Marks	10	20	30	40	50	60
No. of Students	4	7	15	8	7	2

Solution :

Marks	**Frequency**	**Cumulative Frequency**
X	f	cf
10	4	4
20	7	11
30	15	26
40	8	34
50	7	41
60	2	43
		$N = \sum f = 43$

Table : 8.11

$$Q_1 = \text{Size of } \left(\frac{N+1}{4}\right)^{th} \text{ value} = \text{Size } 11^{th} \text{ value} = 20$$

$$Q_3 = \text{Size of } \left(\frac{3(N+1)}{4}\right)^{th} \text{ value} = \text{Size of } 33^{rd} \text{ value} = 40$$

$$QD = \frac{1}{2}(Q_3 - Q_1) = \frac{40-20}{2} = 10$$

$$\text{Coefficient of QD} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{40-20}{40+20} = \frac{20}{60}$$

$$= 0.333$$

Example 8.18

Compute Quartile deviation from the following data

CI	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
f	12	19	5	10	9	6	6

Solution :

CI	f	cf
10 – 20	12	12
20 – 30	19	31
30 – 40	5	36
40 – 50	10	46
50 – 60	9	55
60 – 70	6	61
70 – 80	6	67
	N = 67	

Table : 8.12

$$Q_1 = \text{Size of } \left(\frac{N}{4}\right)^{th} \text{ value } = \left(\frac{67}{4}\right)^{th} = 16.75^{\text{th}} \text{ value}$$

Thus Q_1 lies in the class 20 – 30 ; and corresponding values are

$$L = 20,\ \frac{N}{4} = 16.75\ ;\ pcf = 12,\ f = 19,\ c = 10$$

$$Q_1 = L + \left(\frac{\frac{N}{4} - pcf}{f}\right) \times c$$

$$Q_1 = 20 + \left(\frac{16.75 - 12}{19}\right) \times 10 = 20 + 2.5 = 22.5$$

$$Q_3 = \text{Size of } \left(\frac{3N}{4}\right)^{\text{th}} \text{ value } = 50.25^{\text{th}} \text{ value}$$

Thus Q_3 lies in the class 50 – 60 and corresponding values are

$$L = 50\ ;\ \frac{3N}{4} = 50.25\ ;\ pcf = 46,\ f = 9,\ c = 10$$

$$Q_3 = L + \left(\frac{\frac{3N}{4} - pcf}{f}\right) \times c$$

$$Q_3 = 50 + \left[\frac{50.25 - 46}{9}\right] \times 10 = 54.72$$

$$QD = \frac{1}{2}(Q_3 - Q_1)$$

$$= \frac{54.72 - 22.5}{2} = 16.11$$

$$\therefore \quad QD = 16.11$$

8.2.2 Mean deviation

Mean Deviation (MD) is defined as the average of the absolute difference between the items in a distribution and the mean or median of that series.

(i) Computation of Mean Deviation - Individual observations

If $X_1, X_2, X_3, \ldots X_n$ are n given observation then the mean deviation about mean or median is as follows

$$\text{MD about Mean} = \frac{\sum |X - \bar{X}|}{n} = \frac{\sum |D|}{n}$$

where $|D| = |X - \bar{X}|$ and n is the number of observations.

$$\text{MD about Median} = \frac{\Sigma |x - median|}{n} = \frac{\Sigma |D|}{n}$$

where $|D| = |X - median|$ and n is the number of observations.

NOTE

If the Mean deviation is computed from Median then in that case $|D|$ shall denote deviations of the items from Median, ignoring signs.

(ii) Computation of Mean Deviation - Discrete series

In discrete series the formula for calculating mean deviation is

$$\text{MD about Mean} = \frac{\sum f|X - \bar{X}|}{N} = \frac{\sum f|D|}{N}$$

where $|D| = |X - \bar{X}|$ by ignoring negative signs and N is total frequencies.

$$\text{MD about Median} = \frac{\sum f|X - Median|}{N} = \frac{\sum f|D|}{N}$$

where $|D| = |X - \text{Median}|$ by ignoring negative sign and N is total frequencies.

(iii) Calculation of Mean Deviation- Continuous Series

For calculating Mean deviation in continuous series we have to obtain the midpoints of the various classes and take the deviations of these mid points from mean or median.

$$\text{MD about Mean} = \frac{\sum f|M-\bar{X}|}{N} \quad \text{or}$$

$$\text{MD about Mean} = \frac{\sum f|D|}{N}$$

where M is a mid value, $|D| = |M-\bar{X}|$ and N is the total frequencies.

$$\text{MD about Median} = \frac{\sum f|M-Median|}{N} \quad \text{or}$$

$$\text{MD about Mean} = \frac{\sum f|D|}{N}$$

where M is a mid value, $|D| = |M\text{–Median}|$ (by ignoring negative sign) and N is the total frequencies.

(iv) Relative Measure for Mean Deviation

The relative measure corresponding to the mean deviation is called the coefficient of mean deviation and it is obtained as follows

$$\text{Coefficient of MD about mean} = \frac{Mean\ Deviation\ about\ Mean}{Mean}$$

$$\text{Coefficient of MD about median} = \frac{Mean\ Deviation\ about\ Median}{Median}$$

NOTE

However, in practice the arithmetic mean is more frequently used in calculating the mean deviation. If specifically stated to calculate mean deviation about median, median can be used.

Example 8.19

Calculate the Mean Deviation about mean and its coefficient of the income groups of five, given below

Income Rs	4000	4200	4400	4600	4800

Solution:

Calculation of Mean Deviation about Mean

$$\text{Mean} = \frac{\sum X}{n} = \frac{22000}{5} = 4400$$

Income (Rs) X	$\|D\|=(X-4400)$
4000	400
4200	200
4400	0
4600	200
4800	400
$\sum X = 22000$	$\sum \|D\| = 1200$

Table : 8.13

Median Deviation about Mean $MD = \frac{\Sigma|D|}{n}$;

$$MD = \frac{1200}{5} = 240$$

$$\text{Coefficient of } MD = \frac{240}{4400} = 0.055$$

Example 8.20

Calculate the mean deviation about median and its relative measure for seven numbers given below: 55, 45, 40, 20, 60, 80, and 30.

Solution:

Arrange the values in ascending order 20, 30, 40, 45, 55, 60, 80

$$\text{Median} = \text{size of} \left(\frac{(n+1)}{2}\right)^{th} \text{ value when n is odd}$$

$$= \text{size of} \left(\frac{(7+1)}{2}\right)^{th} \text{ th value}$$

$= \text{size of } 4^{th} \text{ item} = 45$

X	$\|X - \text{Median}\| = \|X - 45\|$
20	25
30	15
40	5
45	0
55	10
60	15
80	30
	$\Sigma \|X - \text{Median}\| = 100$

Table : 8.14

$$\text{MD about Median} = \frac{\sum|X - Median|}{n} = \frac{100}{7} = 14.29$$

$$\text{Coefficient of MD about median} = \frac{14.29}{45} = 0.32$$

Example 8.21

Calculate the Mean deviation about mean for the following data.

Size	2	4	6	8	10	12	14	16
frequency	2	2	4	5	3	2	1	1

Solution:

Calculation of Mean Deviation about Mean

X	f	fX	$\|D\| = X - 8$	$f\|D\|$
2	2	4	6	12
4	2	8	4	8
6	4	24	2	8
8	5	40	0	0
10	3	30	2	6
12	2	24	4	8
14	1	14	6	6
16	1	16	8	8
	$N = 20$	$\Sigma fX = 160$		$\Sigma f\|D\| = 56$

Table : 8.15

$$\overline{X} = \frac{\Sigma fX}{N} = \frac{160}{20} = 8$$

$$\text{Mean Deviation about Mean} = \frac{\Sigma f|D|}{N} = \frac{56}{20} = 2.8$$

Example 8.22

Calculate the Mean deviation about median and its relative measure for the following data.

X	15	25	35	45	55	65	75	85
frequency	12	11	10	15	22	13	18	19

Solution:

Already the values are arranged in ascending order then Median is obtained by the following

X	f	cf
15	12	12
25	11	23
35	10	33
45	15	48
55	22	70
65	13	83
75	18	101
85	19	120
	N = 120	

Table : 8.16

$$\text{Median} = \text{size of} \left(\frac{(n+1)}{2}\right)^{th} \text{value}$$

$$= \text{size of} \left(\frac{(120+1)}{2}\right)^{th} \text{value}$$

$$= \text{size of } 60.5^{\text{th}} \text{ item} = 55$$

$$\text{MD about Median} = \frac{\sum f\left|X - Median\right|}{N} = \frac{\sum f\left|D\right|}{N}$$

Mean deviation about Median

X	f	$\|D\| = \|X - 55\|$	$f\|D\|$
15	12	40	480
25	11	30	330
35	10	20	200
45	15	10	150
55	22	10	220
65	13	10	130
75	18	20	360
85	19	30	570
	N = 120		$\Sigma f\|D\| = 2440$

Table : 8.17

$$\text{MD about Median} = \frac{2440}{120} = 20.33$$

$$\text{Coefficient of mean deviation about median} = \frac{\textit{MD about Median}}{\textit{Median}}$$

$$= \frac{20.33}{55} = 0.37$$

Example 8.23

Find out the coefficient of mean deviation about median in the following series

Age in years	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of persons	20	25	32	40	42	35	10	8

Calculations have to be made correct to two places of decimals.

Solution:

Calculation for median follows by the following table

X	f	cf
0-10	20	20
10-20	25	45
20-30	32	77
30-40	40	117
40-50	42	159
50-60	35	194
60-70	10	204
70-80	8	N=212

Table : 8.18

$\frac{N}{2} = \frac{212}{2} = 106$. Class interval corresponding to cumulative frequency 106 is (30 – 40). So, the corresponding values from the median class are $L = 30$, $pcf = 77$, $f = 40$ and $c = 10$.

$$\text{Median} = L + \left(\frac{\left(\frac{N}{2}\right) - pcf}{f}\right) \times c$$

$$\text{Median} = 30 + \left(\frac{106 - 77}{40}\right) \times 10$$

$\therefore$ Median = 37.25 (corrected to two places of decimals)

Calculations proceeded for mean deviation about the median.

X	f	M	$\lvert D\rvert = \lvert X-37.25\rvert$	$f\lvert D\rvert$
0-10	20	5	32.25	645
10-20	25	15	22.25	556.25
20-30	32	25	12.25	392
30-40	40	35	2.25	90
40-50	42	45	7.75	325.5
50-60	35	55	17.75	621.25
60-70	10	65	27.75	277.5
70-80	8	75	37.75	302
	N=212			$\Sigma f\lvert D\rvert$=3209.5

Table : 8.19

Then the mean deviation about median is to be computed by the following

$$\text{MD about Median} = \frac{\Sigma f\lvert D\rvert}{N} = \frac{3209.5}{212} = 15.14$$

$$\text{Coefficient of MD about Median} = \frac{\text{M.D. about median}}{\text{Median}} = \frac{15.14}{37.25} = 0.4064 = 0.41$$

(corrected to two decimal places).

NOTE

The above problem can also be solved for mean deviation about mean instead of median.

Exercise 8.1

1. Find the first quartile and third quartile for the given observations.

   2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

2. Find Q_1 , Q_3, D_8 and P_{67} of the following data :

Size of Shares	4	4.5	5	5.5	6	6.5	7	7.5	8
Frequency	10	18	22	25	40	15	10	8	7

3. Find lower quartile, upper quartile, 7th decile, 5th decile and 60th percentile for the following frequency distribution.

Wages	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	1	3	11	21	43	32	9

4. Calculate GM for the following table gives the weight of 31 persons in sample survey.

Weight (lbs):	130	135	140	145	146	148	149	150	157
Frequency	3	4	6	6	3	5	2	1	1

5. The price of a commodity increased by 5% from 2004 to 2005 , 8% from 2005 to 2006 and 77% from 2006 to 2007 . Calculate the average increase from 2004 to 2007?

6. An aeroplane flies, along the four sides of a square at speeds of 100,200,300 and 400 kilometres per hour respectively. What is the average speed of the plane in its flight around the square.

7. A man travelled by car for 3 days. He covered 480 km each day. On the first day he drove for 10 hours at 48 km. an hour. On the second day, he drove for 12 hours at 40 km an hour and for the last day he drove for 15 hours at 32 km. What is his average speed?

8. The monthly incomes of 8 families in rupees in a certain locality are given below. Calculate the mean, the geometric mean and the harmonic mean and confirm that the relations among them holds true. Verify their relationships among averages.

Family:	A	B	C	D	E	F	G	H
Income (Rs.):	70	10	50	75	8	25	8	42

9. Calculate AM, GM and HM and also verify their relations among them for the following data

X	5	15	10	30	25	20	35	40
f	18	16	20	21	22	13	12	16

10. Calculate AM, GM and HM from the following data and also find its relationship:

Marks:	0-10	10-20	20-30	30-40	40-50	50-60
No. of students:	5	10	25	30	20	10

11. Calculate the quartile deviation and its coefficient from the following data:

Age in Years:	20	30	40	50	60	70	80
No. of members:	13	61	47	15	10	18	36

12. Calculate quartile deviation and its relative measure from the following data:

X	0-10	10-20	20-30	30-40	40-50	50-60
f	5	10	13	18	14	8

13. Compute mean deviation about median from the following data:

Height in inches	No. of students	Height in inches	No. of students
58	15	63	22
59	20	64	20
60	32	65	10
61	35	66	8
62	35		

14. Compute the mean deviation about mean from the following data:

Class Interval:	0-5	5-10	10-15	15-20	20-25
Frequency *f*	3	5	12	6	4

15. Find out the coefficient of mean deviation about median in the following series

Age in years	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of persons	8	12	16	20	37	25	19	13

8.3 Probability

The word 'probability' or 'chance' is very commonly used in day-to-day conversation and generally people have a rough idea about its meaning. For example, we come across statements like

"Probably it may rain tomorrow";

"The chances of teams A and B winning a certain match are equal";

All these terms – possible, probable, etc., convey the same sense, i.e., the event is not certain to take place or, in other words, there is uncertainty about happening of the event in question. In Layman's terminology the word 'Probability' thus can notes that there is uncertainty about what has happened. However, in mathematics and statistics we try to present conditions under which we can make sensible numerical statements about uncertainty and apply certain methods of calculating numerical values of probabilities.

Galileo (1564-1642), an Italian mathematician, was the first man to attempt quantitative measure of probability while dealing with some problems related to the theory of dice in gambling. The figure given below represents the basic concepts of probability.

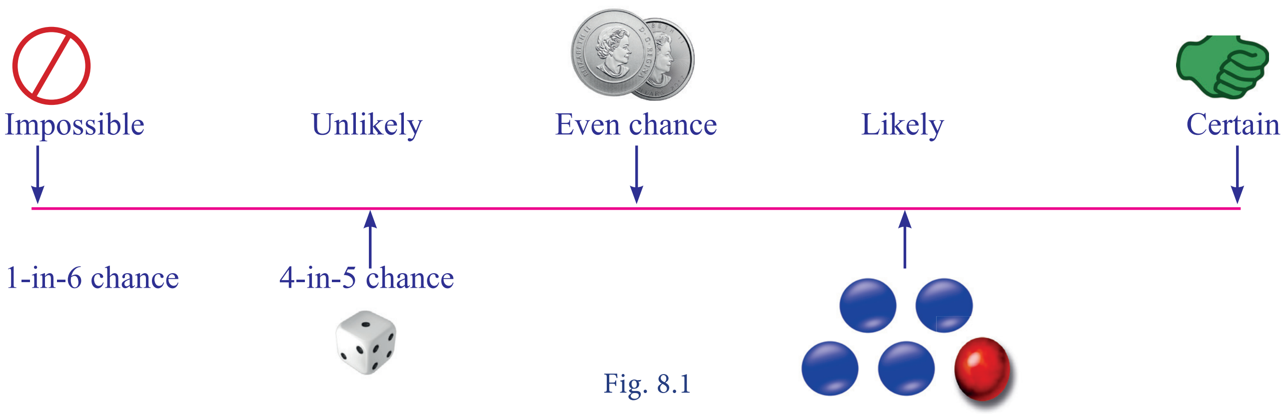


Fig. 8.1

8.3.1 Basic concepts of Probability

Recall

(i) Random Experiment

If an experiment or trial can be repeated under the same conditions, any number of times and it is possible to count the total number of outcomes, but individual result ie., individual outcome is not predictable, then the experiment is known as random experiment.

Example: Tossing a coin, throwing a die, selecting a card from a pack of playing cards, etc.

(ii) Outcome:

The result of a random experiment will be called an outcome.

(iii) Trial and Event:

Any particular performance of a random experiment is called a trial and outcome or combinations of outcomes are termed as events.

(iv) Exhaustive Events:

The total number of possible outcomes of a random experiment is known as the exhaustive events.

(v) Favourable Events:

The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

(vi) Mutually Exclusive events:

Events are said to be mutually exclusive if the happening of any one of them precludes the happening of all the others, ie., if no two or more of them can happen simultaneously in the same trial. Symbolically the event A and B are mutually exclusive if $A \cap B = \varnothing$.

(vii) Equally Likely Events:

Events (two or more) of an experiment are said to be equally likely, if any one of them cannot be expected to occur in preference to the others.

(viii) Classical definition of Probability

If a random experiment or trial results in 'n' exhaustive, mutually exclusive and equally likely outcomes (or cases), out of which m are favourable to the occurrence of an event E, then the probability 'p' of occurrence (or happening) of E, usually denoted by $P(E)$, is given by

DO YOU KNOW?

James Bernoulli who was the first person to obtain a quantitative measure of uncertainty

$$P = P(E) = \frac{Number\ of\ favourable\ cases}{Total\ number\ of\ exhastive\ cases} = \frac{m}{n}$$

(ix) Properties

(i) $0 \leq P(E) \leq 1$

(ii) Sum of all the probability equal to 1.

(iii) If $P(E)=0$ then E is an impossible event.

For example : A coin is tossed. Find the probability of getting a head

Solution: The total possible outcomes of an experiment {H,T}

Therefore $n = 2$

The favourable outcome for getting a head {H}. Therefore m =1. Thus the required probability is

$$P(\text{getting a } H\} = \frac{m}{n} = \frac{1}{2}$$

(x) Modern Definition of Probability

The modern approach to probability is purely axiomatic and it is based on the set theory concepts. In order to study, the theory of probability with an axiomatic approach it is necessary to define certain basic concepts. They are

(i) Sample Space: Each possible outcome of an experiment that can be repeated under similar or identical conditions is called a sample point and the collection of sample points is called the sample space, denoted by S.

(ii) Event: Any subset of a sample space is called an event.

(iii) Mutually Exclusive events: Two events A and B are said to be mutually exclusive events if $A \cap B = \phi$ i.e., if A and B are disjoint sets.

Example: Consider S $= \{1,2,3,4,5\}$

Let A = the set of odd numbers $= \{1,3,5\}$

and B = the set of even numbers $= \{2,4\}$

Then $A \cap B = \phi$

Therefore the events A and B are mutually exclusive

(xi) Observation:

Statement meaning in terms of Set theory approach

(i) $A \cup B \Rightarrow$ at least one of the events A or B occurs

(ii) $A \cap B \Rightarrow$ both events A and B occurs

(iii) $\overline{A} \cap \overline{B} \Rightarrow$ Neither A nor B occur

(iv) $A \cap \overline{B} \Rightarrow$ Event A occurs and B does not occur

(xii) Definition of Probability (Axiomatic approach)

Let E be an experiment. Let S be a sample space associated with E. With every event in S we associate a real number denoted by $P(A)$ called the probability of the event A satisfying the following axioms.

Axiom 1 : $P(A) \geq 0$

Axiom 2 : $P(S) = 1$

Axiom 3 : If $A_1, A_2, \ldots, A_n$ be a sequence of n mutually exclusive events in S then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

(xiii) Basic Theorems on probability

Theorem 1:

Then $P(\varnothing) = 0$ i.e., probability of an impossible event is zero.

Theorem 2:

Let S be the sample space and A be an event in S then $P(\bar{A}) = 1 - P(A)$

Theorem 3: Addition Theorem

If A and B are any two events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(xiv) Observation:

(i) If the two events A and B are mutually exclusive then $A \cap B = \varnothing$

$\therefore \quad P(A \cap B) = 0$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B)$

(ii) The addition theorem may be extended to any three events A, B, C and we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

It is believed that the students might be familiar with the above concepts and our present syllabus continues from the following.

8.3.2 Independent and Dependent events

(i) Independent Events

Two or more events are said to be **independent** when the outcome of one does not affect and is not affected by, the other. For example, if a coin is tossed twice, the result of the second throw would in no way be affected by the result of the first throw.

(ii) Dependent events are those in which the occurrence or non-occurrence of one event in any one trial affects the other events in other trials.

For example the probability of drawing a queen from a pack of 52 cards is $\frac{4}{52}$ or $\frac{1}{13}$. But if the card drawn (queen) is not replaced in the pack, the probability of drawing again a queen is $\frac{3}{51}$.

8.3.3 Conditional Probability

If two events A and B are dependent, then the conditional probability of B given that A as occurred already is

$$P(B/A) = \frac{P(A\cap B)}{P(A)}; \quad P(A) \neq 0$$

Similarly $$P(A/B) = \frac{P(A\cap B)}{P(B)}; \quad P(B) \neq 0$$

(i) Multiplication Theorem:

The probability of the simultaneous happening of two events A and B is given by

$$P(A\cap B) = P(A).P(B/A) \text{ or}$$

$$P(A\cap B) = P(B).P(A/B)$$

NOTE

If A and B are two independent events then $P(A \text{ and } B) = P(A\cap B) = P(A)\ P(B)$

(iii) The theorem can be extended to three or more independent events. Thus for three events the theorem states that

$$P(A \text{ and } B \text{ and } C) = P(A\cap B\cap C) = P(A)\ P(B)P(C)$$

Example 8.24

An unbiased die is thrown. If A is the event 'the number appearing is a multiple of 3' and B be the event 'the number appearing is even' number then find whether A and B are independent?

Solution:

We know that the sample space is $S = \{1,2,3,4,5,6\}$

Now, $A = \{3,6\}$; $B = \{2,4,6\}$ then $(A \cap B) = \{6\}$

$$P(A) = \frac{2}{6} = \frac{1}{3}$$

$$P(B) = \frac{3}{6} = \frac{1}{2} \text{ and } P(A \cap B) = \frac{1}{6}$$

Clearly $P(A \cap B) = P(A)\,P(B)$

Hence A and B are independent events.

Example 8.25

Let $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find $P(A \cap B)$ if A and B are independent events.

Solution:

Since A and B are independent events then $P(A \cap B) = P(A)\ P(B)$

Given that $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$,

then $P(A \cap B) = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$

Example 8.26

Three coins are tossed simultaneously. Consider the events A 'three heads or three tails', B 'atleast two heads' and C 'at most two heads' of the pairs (A,B), (A,C) and (B,C), which are independent? Which are dependent?

Solution:

Here the sample space of the experiment is

$$S = \{HHH, HHT, HTH, HTT, THH, TTH, THT, TTT\}$$

$$A = \{\text{Three heads or Three tails}\} = \{HHH, TTT\}$$

$$B = \{\text{at least two heads}\} = \{HHH, HHT, HTH, THH\}$$

And $C = \{\text{at most two heads}\} = \{HHT, HTH, HTT, THH, TTH, THT, TTT\}$

Also $(A \cap B) = \{HHH\}$; $(A \cap C) = \{TTT\}$ and $(B \cap C) = \{HHT, HTH, THH\}$

$\therefore$ $P(A) = \frac{2}{8} = \frac{1}{4}$; $P(B) = \frac{1}{2}$; $P(C) = \frac{7}{8}$ and

$$P(A \cap B) = \frac{1}{8},\ P(A \cap C) = \frac{1}{8},\ P(B \cap C) = \frac{3}{8}$$

$$\text{Also} \quad P(A).\,P(B) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$$

$$P(A).\,P(C) = \frac{1}{4}\cdot\frac{7}{8} = \frac{7}{32}$$

$$\text{and} \quad P(B).\,P(C) = \frac{1}{2}\cdot\frac{7}{8} = \frac{7}{16}$$

$$\text{Thus,} \quad P(A\cap B) = P(A).\,P(B)$$

$$P(A\cap C) \neq P(A)\,.\,P(C) \text{ and}$$

$$P(B\cap C) \neq P(B).\,P(C)$$

Hence, the events (A and B) are independent, and the events (A and C) and (B and C) are dependent.

Example 8.27

A can solve 90 per cent of the problems given in a book and B can solve 70 per cent. What is the probability that at least one of them will solve a problem selected at random?

Solution:

Given the probability that A will be able to solve the problem $= \frac{90}{100} = \frac{9}{10}$ and the probability that B will be able to solve the problem $= \frac{70}{100} = \frac{7}{10}$

i.e., $P(A) = \frac{9}{10}$ and $P(B) = \frac{7}{10}$

$$P(\bar{A}) = 1 - P(A) = \frac{9}{10} = \frac{1}{10}$$

$$P(\bar{B}) = 1 - P(B) = \frac{7}{10} = \frac{3}{10}$$

$$P(\text{at least one solve the problem}) = P(A\cup B)$$

$$= 1 - P(\overline{A\cup B}) = 1 - P(\bar{A}\cap\bar{B})$$

$$= 1 - P(\bar{A})\cdot P(\bar{B})$$

$$= 1 - \frac{3}{100} = \frac{97}{100}$$

Hence the probability that at least one of them will solve the problem $= \frac{97}{100}$

Example 8.28

A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black.

Solution:

Let A, B be the events of getting a black ball in the first and second draw

Probability of drawing a black ball in the first attempt is

$$P(A) = \frac{3}{5+3} = \frac{3}{8}$$

Probability of drawing the second black ball given that the first ball drawn is black

$$\mathrm{P(B/A)} = \frac{2}{5+2} = \frac{2}{7}$$

$\therefore$ The probability that both balls drawn are black is given by

$$\mathrm{P(A \cap B)} = \mathrm{P(A)\ P(B/A)} = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

Example 8.29

In a shooting test the probability of hitting the target are $\frac{3}{4}$ for A, $\frac{1}{2}$ for B and $\frac{2}{3}$ for C. If all of them fire at the same target, calculate the probabilities that

(i) All the three hit the target

(ii) Only one of them hits the target

(iii) At least one of them hits the target

Solution:

Given $P(\mathrm{A}) = \frac{3}{4}$, $P(\mathrm{B}) = \frac{1}{2}$, $P(\mathrm{C}) = \frac{2}{3}$

Then $P(\overline{A}) = 1 - \frac{3}{4} = \frac{1}{4}$; $P(\overline{B}) = 1 - \frac{1}{2} = \frac{1}{2}$ and $P(\overline{C}) = 1 - \frac{2}{3} = \frac{1}{3}$

(i) $\mathrm{P}(\overline{A})$=(all the three hit the targets)

$$= \mathrm{P(A \cap B \cap C) = P(A)P(B)P(C)} \ \text{(since } A, B, C \text{ hits independently)}$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{4}$$

(ii) P(only one of them hits the target)

$$= P\{(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)\}$$

$$= P\{(A\cap\overline{B}\cap\overline{C})+P(\overline{A}\cap B\cap\overline{C})+P(\overline{A}\cap\overline{B}\cap C)\}$$

$$= \left(\frac{3}{4}.\frac{1}{2}.\frac{1}{3}\right)+\left(\frac{1}{4}.\frac{1}{2}.\frac{1}{3}\right)+\left(\frac{1}{4}.\frac{1}{2}.\frac{2}{3}\right)=\frac{1}{4}$$

(iii) P(at least one of them hit the target)

$= 1-$ P(none of them hit the target)

$$= 1-P(\overline{A}\cap\overline{B}\cap\overline{C})$$

$$= 1-P(\overline{A})P(\overline{B})P(\overline{C})$$

$$= 1-\frac{1}{24}=\frac{23}{24}$$

Example 8.30

Find the probability of drawing a queen, a king and a knave (Jack) in that order from a pack of cards in three consecutive draws, the card drawn not being replaced.

Solution :

Let A : the card drawn is a queen

B: the card drawn is a king

C: the card is drawn is a knave(jack)

P(drawing a queen card) $= P(A) = \frac{4}{52}$

P(drawing a king card given that a queen card has been already drawn)

$= P(B/A) = \frac{4}{51}$

P(drawing a knave card given that a queen and a king cards have been drawn) $= P(C/AB) = \frac{4}{50}$

Since they are dependent events, the required probability of the compound event is

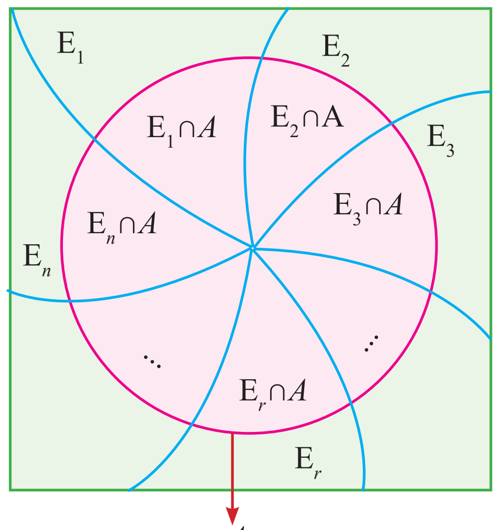
$$P(ABC) = P(A)\ P(B/A)\ P(C/AB) = \frac{4}{52}\times\frac{4}{51}\times\frac{4}{50} = \frac{64}{132600}$$

$$= 0.00048$$

8.3.4 Baye's Theorem

If $E_1, E_2, E_3 ..., E_n$ are a set of n mutually exclusive and collectively exhaustive events with $P(E_i) \neq 0$ ($i = 1,2,3\ldots, n$), then for any arbitrary event A which is associated with sample space $S = \bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i/A)=\frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n}P(E_i)P(A/E_i)}\quad ; \quad i=1,2,3,\ldots,n;$$



where $$P(A)=\sum_{i=1}^{n}P(E_i)P(A/E_i)$$

Example 8.31

Bag I contains 3 red and 4 blue balls while another Bag II contains 5 red and 6 blue balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from second Bag.

Solution:

Let E_1 be the event of choosing the first bag, E_2 the event of choosing the second bag and A be the events of drawing a red ball. Then $P(E_1) = P(E_2) = \frac{1}{2}$

Also $\quad P(A/E_1) = P(\text{drawing a red ball from Bag } I) = \frac{3}{7}$

And $\quad P(A/E_2) = P(\text{drawing a red ball from Bag } II) = \frac{5}{11}$

Now, the probability of drawing a ball from Bag II, being given that it is red, is $P(E_2/A)$.

By using Baye's theorem, we have

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{\sum_{i=1}^{2}P(E_i)P(A/E_i)}$$

$$= \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\cdot\frac{5}{11}}{\left(\frac{1}{2}\times\frac{3}{7}\right)+\left(\frac{1}{2}\times\frac{5}{11}\right)} = \frac{35}{68}$$

Example 8.32

X speaks truth 4 out of 5 times. A die is thrown. He reports that there is a six. What is the chance that actually there was a six?

Solution:

Let us define the following events.

E_1 : X speaks truth;

E_2 : X tells a lie;

E : X reports a six;

From the data given in the problem, we have

$$P(E_1) = \frac{4}{5}\ ; \quad P(E_2) = \frac{1}{5}; \quad P(E/E_1) = \frac{1}{6}; \quad P(E/E_2) = \frac{5}{6}$$

The required probability that actually there was six (by Bayes theorem) is

P(E_1) P(E/E_1)

$$P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)} = \frac{\frac{4}{5}\times\frac{1}{6}}{\left(\frac{4}{5}\times\frac{1}{6}\right)+\left(\frac{1}{5}\times\frac{5}{6}\right)} = \frac{4}{9}$$

P(E_1) P(E/E_1)+ P(E_2) P(E/E_2)

Example 8.33

A factory has 3 machines A_1, A_2, A_3 producing 1000, 2000, 3000 screws per day respectively. A_1 produces 1% defectives, A_2 produces 1.5% and A_3 produces 2% defectives. A screw is chosen at random at the end of a day and found defective. What is the probability that it comes from machines A_1?

Solution:

$$P(A_1) = \text{P(that the machine } A_1 \text{ produces screws)} = \frac{1000}{6000} = \frac{1}{6}$$

$$P(A_2) = \text{P(that the machine } A_2 \text{ produces screws)} = \frac{2000}{6000} = \frac{1}{3}$$

$$P(A_3) = \text{P(that the machine } A_3 \text{ produces screws)} = \frac{3000}{6000} = \frac{1}{2}$$

Let B be the event that the chosen screw is defective

$\therefore \quad P(B/A_1) = \text{P(that defective screw from the machine } A_1) = 0.01$

$P(B/A_2) = \text{P(that defective screw from the machine } A_2) = 0.015$ and

$P(B/A_3) = \text{P(that defective screw from the machine } A_3) = 0.02$

We have to find $P(A_1/B)$

Hence by Baye's theorem, we get

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1)+P(A_2)P(B/A_2)+P(A_3)P(B/A_3)}$$

$$= \frac{\left(\frac{1}{6}\right)(0.01)}{\left(\frac{1}{6}\right)(0.01)+\left(\frac{1}{3}\right)(0.015)+\left(\frac{1}{2}\right)(0.02)}$$

$$= \frac{0.01}{0.01+0.03+0.06} = \frac{0.01}{0.1} = \frac{1}{10}$$

Exercise 8.2

1. A family has two children. What is the probability that both the children are girls given that at least one of them is a girl?

2. A die is thrown twice and the sum of the number appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

3. An unbiased die is thrown twice. Let the event A be odd number on the first throw and B the event odd number on the second throw. Check whether A and B events are independent.

4. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that the problem is
   (i) solved
   (ii) exactly one of them solves the problem

5. Suppose one person is selected at random from a group of 100 persons are given in the following

	Psychologist	Socialist	Democrate	Total
Men	15	25	10	50
Women	20	15	15	50
Total	35	40	25	100

What is the probability that the man selected is a Psychologist?

6. Two urns contains the set of balls as given in the following table

	White	Red	Black
Urn 1	10	6	9
Urn 2	3	7	15

One ball is drawn from each urn and find the probability that

(i) both balls are red

(ii) both balls are of the same colour.

7. Bag I contains 3 Red and 4 Black balls while another Bag II contains 5 Red and 6 Black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag I.

8. The first of three urns contains 7 White and 10 Black balls, the second contains 5 White and 12 Black balls and third contains 17 White balls and no Black ball. A person chooses an urn at random and draws a ball from it. And the ball is found to be White. Find the probabilities that the ball comes from

(i) the first urn

(ii) the second urn

(iii) the third urn

9. Three boxes B_1, B_2, B_3 contain lamp bulbs some of which are defective. The defective proportions in box B_1, box B_2 and box B_3 are respectively $\frac{1}{2}, \frac{1}{8}$ and $\frac{3}{4}$. A box is selected at random and a bulb drawn from it. If the selected bulb is found to be defective, what is the probability that box B_1 was selected?

10. Three horses A, B, C are in race. A is twice as likely to win as B and B is twice as likely to win as C. What are their respective probabilities of winning?

11. A die is thrown. Find the probability of getting
   (i) a prime number
   (ii) a number greater than or equal to 3

12. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 4. What is the probability that it is an even number?

13. There are 1000 students in a school out of which 450 are girls. It is known that out of 450, 20% of the girls studying in class XI. A student is randomly selected from 1000 students. What is the probability that the selected student is from class XI given that the selected student is a girl?

14. From a pack of 52 cards, two cards are drawn at random. Find the probability that one is a king and the other is a queen.

15. A card is drawn from a pack of playing cards and then another card is drawn without the first being replaced. What is the probability of drawing
   (i) two aces
   (ii) two spades

16. A company has three machines A, B, C which produces 20%, 30% and 50% of the product respectively. Their respective defective percentages are 7, 3 and 5. From these products one is chosen and inspected. If it is defective what is the probability that it has been made by machine C?

Exercise 8.3

IKPJ8T

Choose the correct answer:

1. Which of the following is positional measure?

   (a) Range (b) Mode (c) Mean deviation (d) Percentiles

2. When calculating the average growth of economy, the correct mean to use is?

   (a) Weighted mean (b) Arithmetic mean

   (c) Geometric mean (d) Harmonic mean

3. When an observation in the data is zero, then its geometric mean is

(a) Negative (b) Positive

(c) Zero (d) Cannot be calculated

4. The best measure of central tendency is

(a) Arithmetic mean (b) Harmonic mean

(c) Geometric mean (d) Median

5. The harmonic mean of the numbers 2,3,4 is

(a) $\frac{12}{13}$ (b) 12 (c) $\frac{36}{13}$ (d) $\frac{13}{36}$

6. The geometric mean of two numbers 8 and 18 shall be

(a) 12 (b) 13 (c) 15 (d) 11.08

7. The correct relationship among A.M.,G.M.and H.M.is:

(a) $A.M. < G.M. < H.M.$ (b) $G.M. \geq A.M. \geq H.M.$

(c) H.M.≥G.M.≥A.M. (d) $A.M. \geq G.M. \geq H.M.$

8. Harmonic mean is the reciprocal of

(a) Median of the values. (b) Geometric mean of the values.

(c) Arithmetic mean of the reciprocal of the values. (d) Quartiles of the values.

9. Median is same as

(a) Q_1 (b) Q_2 (c) Q_3 (d) D_2

10. The median of 10,14,11,9,8,12,6 is

(a) 10 (b) 12 (c) 14 (d) 9

11. The mean of the values 11,12,13,14 and 15 is

(a) 15 (b) 11 (c) 12.5 (d) 13

12. If the mean of 1,2,3, …, n is $\frac{6n}{11}$, then the value of n is

(a) 10 (b) 12 (c) 11 (d) 13

13. Harmonic mean is better than other means if the data are for

(a) Speed or rates. (b) Heights or lengths.
(c) Binary values like 0 and 1. (d) Ratios or proportions.

14. The first quartile is also known as

(a) median. (b) lower quartile. (c) mode. (d) third decile

15. If $Q_1 = 30$ and $Q_3 = 50$, the coefficient of quartile deviation is

(a) 20 (b) 40 (c) 10 (d) 0.25

16. If median = 45 and its coefficient is 0.25, then the mean deviation about median is

(a) 11.25 (b) 180 (c) 0.0056 (d) 45

17. The two events A and B are mutually exclusive if

(a) $P(A\cap B)=0$ (b) $P(A\cap B)=1$ (c) $P(A\cup B)=0$ (d) $P(A\cup B) = 1$

18. The events A and B are independent if

(a) $P(A\cap B)=0$ (b) $P(A\cap B)=P(A)\times P(B)$
(c) $P(A\cap B)=P(A)+P(B)$ (d) $P(A\cup B)= P(A)\times P(B)$

19. If two events A and B are dependent then the conditional probability of $P(B/A)$ is

(a) $P(A)\ P(B/\mathrm{A})$ (b) $\dfrac{P(A\cap B)}{P(B)}$
(c) $\dfrac{P(A\cap B)}{P(A)}$ (d) $P(A)\ P(A/B)$

20. The probability of drawing a spade from a pack of card is

(a) 1/52 (b) 1/13 (c) 4/13 (d) 1/4

21. If the outcome of one event does not influence another event then the two events are

(a) Mutually exclusive (b) Dependent
(c) Not disjoint (d) Independent

22. Let a sample space of an experiment be $S=\{E_1,E_2,\ldots,E_n\}$ then $\sum_{i=1}^{n}P(E_i)$ is equal to

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

23. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 1/36 (b) 0 (c) 1/3 (d) 1/6

24. Probability of an impossible event is

(a) 1 (b) 0 (c) 0.2 (d) 0.5

25. Probability that at least one of the events A, B occur is

(a) $P(A \cup B)$ (b) $P(A \cap B)$ (c) $P(A/B)$ (d) $(A \cup B)$

Miscellaneous Problems

1. Find out the GM for the following

Yield of Rice (tones)	No. of farms
7.5-10.5	5
10.5-13.5	9
13.5-16.5	19
16.5-19.5	23
19.5-22.5	7
22.5-25.5	4
25.5-28.5	1

2. An investor buys Rs. 1,500 worth of shares in a company each month. During the first four months he bought the shares at a price of Rs. 10, 15, 20 and 30 per share. What is the average price paid for the shares bought during these four months? Verify your result.

3. Calculate Mean deviation about median of the following data .

Class interval :	0-10	10-20	20-30	30-40	40-50	50-60
Frequency :	6	7	15	16	4	2

4. Calculate Mean deviation about Mean of the following data.

X	2	5	6	8	10	12
f	2	8	10	7	8	5

5. Calculate Quartile deviation and Coefficient of Quartile deviation of the following data.

Marks:	0	10	20	30	40	50	60	70
No. of students:	150	142	130	120	72	30	12	4

6. In a screw factory machines A, B, C manufacture respectively 30%, 40% and 30% of the total output of these 2% , 4% and 6% percent are defective screws. A screws is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by Machine C?

7. A committee of two persons is formed from 3 men and 2 women. What is the probability that the committee will have

   (i) No woman (ii) One man (iii) No man

8. A, B and C was 50% , 30% and 20% of the cars in a service station respectively. They fail to clean the glass in 5% , 7% and 3% of the cars respectively. The glass of a washed car is checked. What is the probability that the glass has been cleaned?

9. Data on readership of a magazine indicates that the proportion of male readers over 30 years old is 0.30 and the proportion of male reader under 30 is 0.20. If the proportion of readers under 30 is 0.80. What is the probability that a randomly selected male subscriber is under 30?

10. Gun 1 and Gun 2 are shooting at the same target. Gun 1 shoots on the average nine shots during the same time Gun 2 shoots 10 shots. The precision of these two guns is not the same. On the average, out of 10 shots from Gun 2 seven hit the target. In the course of shooting the target has been hit by a bullet, but it is not known which Gunshot this bullet. Find the chance that the target was hit by Gun 2?

Summary

- A measure which divides an array into four equal parts in known as **quartiles**.
- A measure which divides an array into ten equal parts is known as **deciles**.
- A measure which divides an array into hundred equal parts is known as **percentiles**
- $Q_2 = D_5 = P_{50} =$ Median
- Inter quartile range $= Q_3 - Q_1$
- QD $= \dfrac{Q_3 - Q_1}{2}$

- Harmonic mean = $\dfrac{n}{\dfrac{1}{X_1}+\dfrac{1}{X_2}+\dfrac{1}{X_3}+\cdots\dfrac{1}{X_n}} = \dfrac{n}{\sum\dfrac{1}{X}}$

- Mean Deviation for Individual series MD = $\dfrac{\sum|X-\bar{X}|}{n} = \dfrac{\sum|D|}{n}$

- The conditional probability of an event A given the occurrence of the event B is given by

$$P(A|B) = \frac{P(A\cap B)}{P(B)},\ P(B)\neq 0$$

- Baye's Theorem:

If $E_1, E_2, E_3, \ldots, E_n$ are a set of n mutually exclusive and collectively exhaustive events with $P(E_i)\neq 0$ (i =1,2,3…,n), then for any arbitrary event A which is associated with sample space $S=\bigcup_{i=1}^{n}E_i$ such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n}P(E_i)P(A/E_i)}\ ;\quad i=1,2,3\ldots,\ n\ ;$$

where $\quad P(A) = P(A) = \sum_{i=1}^{n}P(E_i)P(A/E_i)$

GLOSSARY	
Conditional probability	நிபந்தனைக்குட்பட்ட நிகழ்தகவு
Continuous series	தொடர்ச்சியான தொடர்
Decile	பதின்மானம்
Dependent events	சார்பு நிகழ்வுகள்
Discrete series	தனித்த தொடர்
Equally likely events	சம வாய்ப்புள்ள நிகழ்வுகள்
Exhaustive events	முழுமையான நிகழ்வுகள்
Frequency	அலைவெண் / நிகழ்வெண்
Grouped data	தொகுக்கப்பட்ட விவரங்கள்
Independent events	சார்பில்லா நிகழ்வுகள்
Mean deviation	சராசரி விலக்கம்
Mode	முகடு
Mutually exclusive events/ disjoint events	ஒன்றை ஒன்று விலக்கும் நிகழ்வுகள்
Percentile	நூற்றுமானம்/ சதமானம்
Probability	நிகழ்தகவு
Quartile	கால்மானம்
Quartile deviation	கால்மான விலக்கம்
Random experiment	சமவாய்ப்பு சோதனை
Range	வீச்சு
Sample space	கூறுவெளி

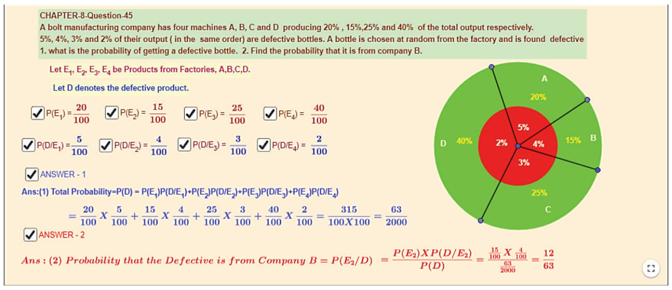
ICT Corner

Descriptive statistics and probability

Step – 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "11th Business Maths Volume-2" will appear. In that there are several worksheets related to your Text Book.



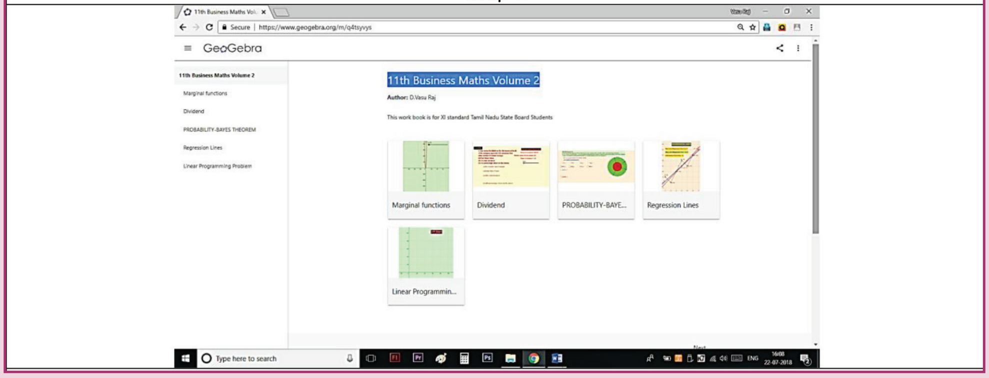
Expected Outcome ⇒

Step-2

Select the work sheet "Probability-Bayes theorem" Find each probabilities step by step as shown and Click on the respective boxes to see the answers.

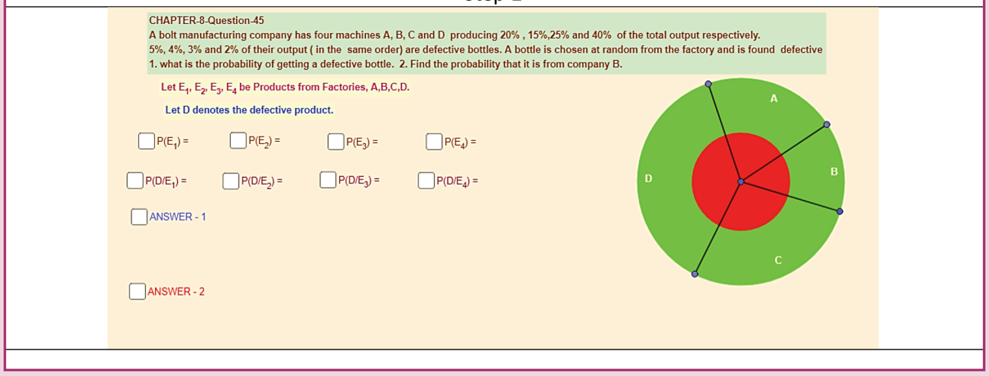
Step 1

Step-1



Step 2

Step-2



Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

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Chapter 9

Correlation and Regression analysis

Learning Objectives

After studying this chapter students will be able to understand

- Concept of Karl Pearson's correlation co-efficient and the methods of computing it.
- Spearman's Rank correlation co-efficient
- Concept of Regression and Regression co-efficients
- Regression lines both x on y and y on x

IKYFAG

9.1 Correlation

Introduction

In the previous Chapter we have studied the characteristics of only one variable; example, marks, weights, heights, rainfalls, prices, ages, sales, etc. This type of analysis is called univariate analysis. Sometimes we may be interested to find if there is any relationship between the two variables under study. For example, the price of the commodity and its sale, height of a father and height of his son, price and demand, yield and rainfall, height and weight and so on. Thus the association of any two variables is known as correlation.

Karl Pearson

Correlation is the statistical analysis which measures and analyses the degree or extent to which two variables fluctuate with reference to each other.

9.1.1 Meaning of Correlation

The term correlation refers to the degree of relationship between two or more variables. If a change in one variable effects a change in the other variable, the variables are said to be correlated.

9.1.2 Types of correlation

Correlation is classified into many types, but the important are:

(i) Positive

(ii) Negative

Positive and negative correlation depends upon the direction of change of the variables.

Positive Correlation

If two variables tend to move together in the same direction that is, an increase in the value of one variable is accompanied by an increase in the value of the other variable; or a decrease in the value of one variable is accompanied by a decrease in the value of the other variable, then the correlation is called positive or direct correlation.

Example

(i) The heights and weights of individuals

(ii) Price and Supply

(iii) Rainfall and Yield of crops

(iv) The income and expenditure

Negative Correlation

If two variables tend to move together in opposite direction so that an increase or decrease in the values of one variable is accompanied by a decrease or increase in the value of the other variable, then the correlation is called negative or inverse correlation.

Example

(i) Price and demand

(ii) Repayment period and EMI

(iii) Yield of crops and price

No Correlation

Two variables are said to be uncorrelated if the change in the value of one variable has no connection with the change in the value of the other variable.

For example

We should expect zero correlation (no correlation) between weight of a person and the colour of his hair or the height of a person and the colour of his hair.

Simple correlation

The correlation between two variables is called simple correlation. The correlation in the case of more than two variables is called multiple correlation.

The following are the mathematical methods of correlation coefficient

(i) Scatter diagram

(ii) Karl Pearson's Coefficient of Correlation

9.1.3 Scatter Diagram

Let $(X_1, Y_1), (X_2, Y_2) \ldots (X_N, Y_N)$ be the N pairs of observation of the variables X and Y. If we plot the values of X along x - axis and the corresponding values of Y along y-axis, the diagram so obtained is called a scatter diagram. It gives us an idea of relationship between X and Y. The type of scatter diagram under a simple linear correlation is given below.

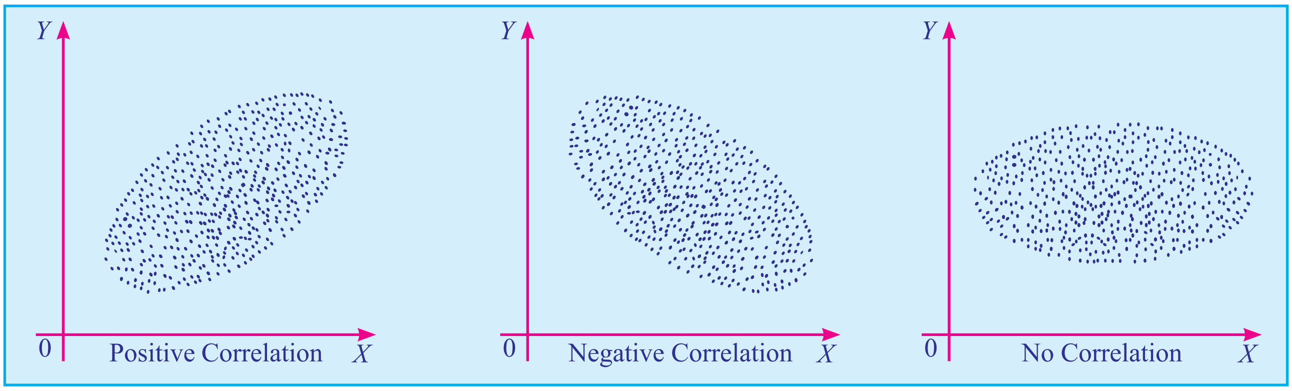


Fig 9.1

(i) If the plotted points show an upward trend, the correlation will be positive.

(ii) If the plotted points show a downward trend, the correlation will be negative.

(iii) If the plotted points show no trend the variables are said to be uncorrelated.

9.1.4 Karl Pearson's Correlation Coefficient

Karl Pearson, a great biometrician and statistician, suggested a mathematical method for measuring the magnitude of linear relationship between two variables say X and Y. Karl Pearson's method is the most widely used method in practice and is known as Pearsonian Coefficient of Correlation. It is denoted by the symbol 'r' and defined as

$$r = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}, \text{ where cov}(X, Y) = \frac{1}{N}\sum(X-\overline{X})(Y-\overline{Y})$$

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(X_i-\overline{X})^2}$$

$$\sigma_y = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(Y_i-\overline{Y})^2}$$

Hence the formula to compute Karl Pearson Correlation coefficient is

$$r = \frac{\frac{1}{N}\sum_{i=1}^{N}(X_i-\overline{X})(Y_i-\overline{Y})}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}(X_i-\overline{X})^2}\sqrt{\frac{1}{N}\sum_{i=1}^{N}(Y_i-\overline{Y})^2}}$$

$$r = \frac{\sum_{i}(X_i-\overline{X})(Y_i-\overline{Y})}{\sqrt{\sum_{i=1}^{N}(X_i-\overline{X})^2}\sqrt{\sum_{i=1}^{N}(Y_i-\overline{Y})^2}}$$

Interpretation of Correlation coefficient:

Coefficient of correlation lies between −1 and +1. Symbolically, $-1 \le r \le +1$

- When $r = +1$, then there is perfect positive correlation between the variables.
- When $r = -1$, then there is perfect negative correlation between the variables.
- When $r = 0$, then there is no relationship between the variables, that is the variables are uncorrelated.

Thus, the coefficient of correlation describes the magnitude and direction of correlation.

Methods of computing Correlation Coefficient

(i) When deviations are taken from Mean

Of all the several mathematical methods of measuring correlation, the Karl Pearson's method, popularly known as Pearsonian coefficient of correlation, is most widely used in practice.

$$r = \frac{\sum_{i}(X_i-\overline{X})(Y_i-\overline{Y})}{\sqrt{\sum_{i=1}^{N}(X_i-\overline{X})^2}\sqrt{\sum_{i=1}^{N}(Y_i-\overline{Y})^2}} = \frac{\sum xy}{\sqrt{\sum x^2 \sum y^2}}$$

Where $x = (X_i - \overline{X})$ and $y = (Y_i - \overline{Y})$; $i = 1, 2 \ldots N$

This method is to be applied only when the deviations of items are taken from actual means.

Steps to solve the problems:

(i) Find out the mean of the two series that is $\bar{X}$ and $\bar{Y}$.

(ii) Take deviations of the two series from $\bar{X}$ and $\bar{Y}$ respectively and denoted by x and y.

(iii) Square the deviations and get the total of the respective squares of deviation of x and y respectively and it is denoted by Σx^2 and Σy^2 .

(iv) Multiply the deviations of x and y and get the total and it is denoted by Σxy.

(v) Substitute the values of Σxy, Σx^2 and Σy^2 in the above formula.

Example 9.1

Calculate Karl Pearson's coefficient of correlation from the following data:

X:	6	8	12	15	18	20	24	28	31
Y:	10	12	15	15	18	25	22	26	28

Solution:

X	$x = (X-18)$	x^2	Y	$y = (Y-19)$	y^2	xy
6	−12	144	10	−9	81	108
8	−10	100	12	−7	49	70
12	−6	36	15	−4	16	24
15	−3	9	15	−4	16	12
18	0	0	18	−1	1	0
20	2	4	25	6	36	12
24	6	36	22	3	9	18
28	10	100	26	7	49	70
31	13	169	28	9	81	117
$\sum X = 162$	$\sum x = 0$	$\sum x^2 = 598$	$\sum Y = 171$	$\sum y = 0$	$\sum y^2 = 338$	$\sum xy = 431$

Table 9.1

$$N=9,\ \overline{X} = \frac{\Sigma X}{N} = \frac{162}{9} = 18,\ \overline{Y} = \frac{\Sigma Y}{N} = \frac{171}{9} = 19$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}}$$

where $x=(X-\overline{X})$ and $y=(Y-\overline{Y})$

$$\Sigma xy = 431,\ \Sigma x^2 = 598,\ \Sigma y^2 = 338$$

$$r = \frac{431}{\sqrt{598 \times 338}} = \frac{431}{449.582} = +0.959$$

(ii) When actual values are taken (without deviation)

when the values of X and Y are considerably small in magnitude the following formula can be used

$$r = \frac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{N\Sigma X^2 - (\Sigma X)^2} \times \sqrt{N\Sigma Y^2 - (\Sigma Y)^2}}$$

Example 9.2

Calculate coefficient of correlation from the following data

X	12	9	8	10	11	13	7
Y	14	8	6	9	11	12	3

Solution:

In both the series items are in small number. Therefore correlation coefficient can also be calculated without taking deviations from actual means or assumed mean.

X	Y	X^2	Y^2	XY
12	14	144	196	168
9	8	81	64	72
8	6	64	36	48
10	9	100	81	90
11	11	121	121	121
13	12	169	144	156
7	3	49	9	21
$\sum X = 70$	$\sum Y = 63$	$\sum X^2 = 728$	$\sum Y^2 = 651$	$\sum XY = 676$

Table 9.2

$$r = \frac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{N\Sigma X^2 - (\Sigma X)^2} \times \sqrt{N\Sigma Y^2 - (\Sigma Y)^2}}$$

$$= \frac{7(676)-(70)(63)}{\sqrt{7(728)-(70)^2} \times \sqrt{7(651)-(63)^2}}$$

$$= \frac{322}{339.48}$$

$$r = +0.95$$

(iii) When deviations are taken from an Assumed mean

When actual means are in fractions, say the actual means of X and Y series are 20.167 and 29.23, the calculation of correlation by the method discussed above would involve too many calculations and would take a lot of time. In such cases we make use of the assumed mean method for finding out correlation. When deviations are taken from an assumed mean the following formula is applicable:

$$r = \frac{N\Sigma\, dx\, dy - (\Sigma dx)(\Sigma dy)}{\sqrt{N\Sigma dx^2 - (\Sigma dx)^2} \times \sqrt{N\Sigma dy^2 - (\Sigma dy)^2}}$$

Where $dx = X-A$ and $dy=Y-B$. Here A and B are assumed mean

NOTE

While applying assumed mean method, any value can be taken as the assumed mean and the answer will be the same. However, the nearer the assumed mean to the actual mean, the lesser will be the calculations.

Steps to solve the problems:

(i) Take the deviations of X series from an assumed mean, denote these deviations by dx and obtain the total that is Σdx.

(ii) Take the deviations of Y series from an assumed mean, denote these deviations by dy and obtain the total that is Σdy.

(iii) Square dx and obtain the total Σdx^2.

(iv) Square dy and obtain the total Σdy^2.

(v) Multiply dx and dy and obtain the total $\Sigma dx\, dy$

(vi) Substitute the values of $\Sigma\, dxdy$, $\Sigma\, dx$, $\Sigma\, dy$, $\Sigma\, dx^2$ and $\Sigma\, dy^2$ in the formula given above.

Example 9.3

Find out the coefficient of correlation in the following case and interpret.

Height of father (in inches)	65	66	67	67	68	69	71	73
Height of son (in inches)	67	68	64	68	72	70	69	70

Solution:

Let us consider Height of father (in inches) is represented as X and Height of son (in inches) is represented as Y

X	$dx=(X-67)$	dx^2	Y	$dy=(Y-68)$	dy^2	$dxdy$
65	-2	4	67	–1	1	2
66	-1	1	68	0	0	0
67	0	0	64	-4	16	0
67	0	0	68	0	0	0
68	1	1	72	4	16	4
69	2	4	70	2	4	4
71	4	16	69	1	1	4
73	6	36	70	2	4	12
$\sum X=546$	$\sum dx=10$	$\sum dx^2=62$	$\sum Y=548$	$\sum dy=4$	$\sum dy^2=42$	$\sum dxdy=26$

Table 9.3

$$r = \frac{N\,\Sigma dx\,dy-(\Sigma dx)(\Sigma dy)}{\sqrt{N\Sigma dx^2-(\Sigma dx)^2}\times\sqrt{N\Sigma dy^2-(\Sigma dy)^2}}$$

Where

$\Sigma dx = 10,\ \Sigma dx^2 = 62,\ \Sigma dy = 4,\ \Sigma dy^2 = 42$ and $\Sigma dxdy = 26$

$$r = \frac{(8\times 26)-(10\times 4)}{\sqrt{(8\times 62)-(10)^2}\times\sqrt{(8\times 42)-(4)^2}}$$

$$r = \frac{168}{\sqrt{396}\times\sqrt{320}}$$

$$r = \frac{168}{355.98} = 0.472$$

$$r = +0.472$$

Heights of fathers and their respective sons are positively correlated.

Example 9.4

Calculate the correlation coefficient from the following data

$N=9, \sum X=45, \quad \sum Y=108, \quad \Sigma X^2=285, \quad \Sigma Y^2=1356, \quad \Sigma XY=597$

Solution

We know that correlation coefficient

$$r = \frac{N\Sigma XY-(\Sigma X)(\Sigma Y)}{\sqrt{N\Sigma X^2-(\Sigma X)^2}\sqrt{N\Sigma Y^2-(\Sigma Y)^2}}$$

$$= \frac{9(597)-(45\times 108)}{\sqrt{9(285)-(45)^2}\times\sqrt{9(1356)-(108^2)}}$$

$$r = +0.95$$

Example 9.5

From the following data calculate the correlation coefficient $\Sigma xy=120, \Sigma x^2=90, \Sigma y^2=640$

Solution:

Given $\Sigma xy=120, \ \Sigma x^2=90, \ \Sigma y^2=640$

Then $$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}} = \frac{120}{\sqrt{90(640)}} = \frac{120}{\sqrt{57600}} = \frac{120}{240} = 0.5$$

$\therefore$ $$r = 0.5$$

9.2 Rank correlation

9.2.1 Spearman's Rank Correlation Coefficient

In 1904, Charles Edward Spearman, a British psychologist found out the method of ascertaining the coefficient of correlation by ranks. This method is based on rank. This measure is useful in dealing with qualitative characteristics, such as intelligence, beauty, morality, character, etc. It cannot be measured quantitatively, as in the case of Pearson's coefficient of correlation.

Rank correlation is applicable only to individual observations. The result we get from this method is only an approximate one, because under ranking method original value are not taken into account. The formula for Spearman's rank correlation which is denoted by ρ (pronounced as row) is

$$\rho = 1-\frac{6\Sigma d^2}{N(N^2-1)} \quad \text{or}$$

$$\rho = 1 - \frac{6\Sigma d^2}{N^3 - N}$$

where d = The difference of two ranks = $R_X - R_Y$ and

N = Number of paired observations.

Rank coefficient of correlation value lies between -1 and $+1$. Symbolically, $-1 \leq \rho \leq +1$

When we come across spearman's rank correlation, we may find two types of problem

(i) When ranks are given

(ii) When ranks are not given

Example 9.6

The following are the ranks obtained by 10 students in Statistics and Mathematics

Statistics	1	2	3	4	5	6	7	8	9	10
Mathematics	1	4	2	5	3	9	7	10	6	8

Find the rank correlation coefficient.

Solution:

Let R_X is considered for the ranks of Statistics and R_Y is considered for the ranks of mathematics.

R_X	R_Y	$d = R_X - R_Y$	d^2
1	1	0	0
2	4	-2	4
3	2	1	1
4	5	-1	1
5	3	2	4
6	9	-3	9
7	7	0	0
8	10	-2	4
9	6	3	9
10	8	2	4
			$\sum d^2 = 36$

Table 9.4

The rank correlation is given by

$$\rho = 1 - \frac{6\Sigma d^2}{N(N^2-1)} = 1 - \frac{6(36)}{10(10^2-1)}$$

$$= 1 - 0.218$$

$$\therefore \qquad \rho = 0.782$$

Example 9.7

Ten competitors in a beauty contest are ranked by three judges in the following order

First judge	1	4	6	3	2	9	7	8	10	5
Second judge	2	6	5	4	7	10	9	3	8	1
Third judge	3	7	4	5	10	8	9	2	6	1

Use the method of rank correlation coefficient to determine which pair of judges has the nearest approach to common taste in beauty?

Solution:

Let R_X, R_Y, R_Z denote the ranks by First judge, Second judge and third judge respectively

R_X	R_Y	R_Z	$d_{XY} = R_X - R_Y$	$d_{YZ} = R_Y - R_Z$	$d_{ZX} = R_Z - R_X$	d_{XY}^2	d_{YZ}^2	d_{ZX}^2
1	2	3	-1	-1	2	1	1	4
4	6	7	-2	-1	3	4	1	9
6	5	4	1	1	-2	1	1	4
3	4	5	-1	-1	2	1	1	4
2	7	10	-5	-3	8	25	9	64
9	10	8	-1	2	-1	1	4	1
7	9	9	-2	0	2	4	0	4
8	3	2	5	1	-6	25	1	36
10	8	6	2	2	-4	4	4	16
5	1	1	4	0	-4	16	0	16
						$\sum d_{XY}^2 = 82$	$\sum d_{YZ}^2 = 22$	$\sum d_{ZX}^2 = 158$

Table 9.5

$$\rho_{XY} = 1 - \frac{6\Sigma d_{XY}^2}{N(N^2-1)} = 1 - \frac{6(82)}{10(10^2-1)} = 1 - 0.4969 = 0.5031$$

$$\rho_{YZ} = 1 - \frac{6\Sigma d_{YZ}^2}{N(N^2-1)} = 1 - \frac{6(22)}{10(10^2-1)} = 1 - \frac{132}{990}$$

$$= 1 - 0.1333 = 0.8667$$

$$\rho_{ZX} = 1 - \frac{6\Sigma d_{ZX}^2}{N(N^2-1)} = 1 - \frac{6(158)}{10(10^2-1)}$$

$$= 1 - 0.9576 = 0.0424$$

Since the rank correlation coefficient between Second and Third judges i.e., ρ_{YZ} is positive and weight among the three coefficients. So, Second judge and Third judge have the nearest approach for common taste in beauty.

Example 9.8

Calculate rank correlation coefficient of the following data

Subject 1	40	46	54	60	70	80	82	85	87	90	95
Subject2	45	46	50	43	40	75	55	72	65	42	70

Solution:

Let X is considered for Subject1 and Y is considered for Subject2

X	Y	R_X	R_Y	$d = R_X - R_Y$	d^2
40	45	1	4	–3	9
46	46	2	5	–3	9
54	50	3	6	–3	9
60	43	4	3	1	1
70	40	5	1	4	16
80	75	6	11	–5	25
82	55	7	7	0	0
85	72	8	10	–2	4
87	65	9	8	1	1
90	42	10	2	8	64
95	70	11	9	2	4
					$\sum d^2 = 142$

Table 9.6

$$\rho = 1 - \frac{6\Sigma d^2}{N(N^2-1)}$$

$$\rho = 1 - \frac{6(142)}{11(11^2-1)}$$

$$\rho = 1 - \frac{852}{1320} = 0.354$$

Exercise 9.1

1. Calculate the correlation co-efficient for the following data

X	5	10	5	11	12	4	3	2	7	1
Y	1	6	2	8	5	1	4	6	5	2

2. Find coefficient of correlation for the following:

Cost (Rs.)	14	19	24	21	26	22	15	20	19
Sales (Rs.)	31	36	48	37	50	45	33	41	39

3. Calculate coefficient of correlation for the ages of husbands and their respective wives:

Age of husbands	23	27	28	29	30	31	33	35	36	39
Age of wives	18	22	23	24	25	26	28	29	30	32

4. Calculate the coefficient of correlation between X and Y series from the following data

	X	Y
Number of pairs of observation	15	15
Arithmetic mean	25	18
Standard deviation	3.01	3.03
Sum of squares of deviation from the arithmetic mean	136	138

Summation of product deviations of X and Y series from their respective arithmetic means is 122.

5. Calculate correlation coefficient for the following data

X	25	18	21	24	27	30	36	39	42	48
Y	26	35	48	28	20	36	25	40	43	39

6. Find coefficient of correlation for the following:

X	78	89	96	69	59	79	68	62
Y	121	72	88	60	81	87	123	92

7. An examination of 11 applicants for a accountant post was taken by a finance company. The marks obtained by the applicants in the reasoning and aptitude tests are given below.

Applicant	A	B	C	D	E	F	G	H	I	J	K
Reasoning test	20	50	28	25	70	90	76	45	30	19	26
Aptitude test	30	60	50	40	85	90	56	82	42	31	49

Calculate Spearman's rank correlation coefficient from the data given above.

8. The following are the ranks obtained by 10 students in commerce and accountancy are given below

Commerce	6	4	3	1	2	7	9	8	10	5
Accountancy	4	1	6	7	5	8	10	9	3	2

To what extent is the knowledge of students in the two subjects related?

9. A random sample of recent repair jobs was selected and estimated cost and actual cost were recorded.

Estimated cost	300	450	800	250	500	975	475	400
Actual cost	273	486	734	297	631	872	396	457

Calculate the value of spearman's correlation coefficient.

10. The rank of 10 students of same batch in two subjects A and B are given below. Calculate the rank correlation coefficient.

Rank of A	1	2	3	4	5	6	7	8	9	10
Rank of B	6	7	5	10	3	9	4	1	8	2

9.3 Regression Analysis

Introduction:

So far we have studied correlation analysis which measures the direction and strength of the relationship between two variables. Here we can estimate or predict the value of one variable from the given value of the other variable. For instance, price and supply are correlated. We can find out the expected amount of supply for a given price or the required price level for attaining the given amount of supply.

The term " regression" literally means " Stepping back towards the average". It was first used by British biometrician Sir Francis Galton (1822 -1911), in connection with the inheritance of stature. Galton found that the offsprings of abnormally tall or short parents tend to "regress" or "step back" to the average population height. But the term "regression" as now used in Statistics is only a convenient term without having any reference to biometry.

Definition 9.1

Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of the data.

9.3.1 Dependent and independent variables

Definition 9.2

In regression analysis there are two types of variables. The variable whose value is to be predicted is called dependent variable and the variable which is used for prediction is called independent variable

Regression helps us to estimate the value of one variable, provided the value of the other variable is given. The statistical method which helps us to estimate the unknown value of one variable from the known value of the related variable is called Regression.

9.3.2 Regression Equations

Regression equations are algebraic expressions of the regression lines. Since there are two regression lines, there are two regression equations. The regression equation of X on Y is used to describe the variation in the values of X for given changes in Y and the regression equation of Y on X is used to describe the variation in the values of Y for given changes in X. Regression equations of (i) X on Y (ii) Y on X and their coefficients in different cases are described as follows.

Case 1: When the actual values are taken

When we deal with actual values of X and Y variables the two regression equations and their respective coefficients are written as follows

(i) **Regression Equation of X on Y:** $X - \overline{X} = b_{xy}(Y - \overline{Y})$

where ($\overline{X}$)is the mean of X series,

$\overline{Y}$ is the mean of Y series,

$b_{xy} = r\dfrac{\sigma_x}{\sigma_y} = \dfrac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{N\Sigma Y^2 - (\Sigma Y)^2}$ is known as the regression coefficient of X on Y, and r is the correlation coefficient between X and Y, σ_x and σ_y are standard deviations of X and Y respectively.

(ii) Regression Equation of *Y* on X; $Y - \overline{Y} = b_{yx}(X - \overline{X})$

where $\overline{X}$ is the mean of X series,

$\overline{Y}$ is the mean of Y series,

$b_{yx} = r\dfrac{\sigma y}{\sigma x} = \dfrac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{N\Sigma Y^2 - (\Sigma Y)^2}$ is known as the regression coefficient of Y on X, and r is the correlation coefficient between X and Y, σ_x and σ_y are standard deviations of X and Y respectively.

Case 2 Deviations taken from Arithmetic means of X and Y

The calculation can very much be simplified instead of dealing with the actual values of X and Y, we take the deviations of X and Y series from their respective means. In such a case the two regression equations and their respective coefficients are written as follows:

(i) Regression Equation of X on Y: $X - \overline{X} = b_{xy}\ (Y - \overline{Y})$

where ($\overline{X}$)is the mean of X series,

$\overline{Y}$ is the mean of Y series,

$$b_{xy} = r\frac{\sigma x}{\sigma y} = \frac{\Sigma xy}{\Sigma y^2}$$ is known as the regression coefficient of X on Y,

$x = (X - \overline{X})$ and $y = (Y - \overline{Y})$

(ii) Regression Equation of Y on X;

$Y - \overline{Y} = b_{yx}(X - \overline{X})$

where $\overline{X}$ is the mean of X series,

$\overline{Y}$ is the mean of Y series,

$$b_{yx} = r\frac{\sigma y}{\sigma x} = \frac{\Sigma xy}{\Sigma x^2}$$ is known as the regression coefficient of Y on X,

$x = (X - \overline{X})$ and $y = (Y - \overline{Y})$

Note: Instead of finding out the values of correlation coefficient σ_x, σ_y, etc, we can find the value of regression coefficient by calculating Σxy and Σy^2

Case 3 Deviations taken from Assumed Mean

When actual means of X and Y variables are in fractions the calculations can be simplified by taking the deviations from the assumed means. The regression equations and their coefficients are written as follows

(i) Regression Equation of Y on X

$$Y - \overline{Y} = b_{yx}(X - \overline{X})$$

$$b_{yx} = \frac{N\Sigma dxdy - (\Sigma dx)(\Sigma dy)}{N\Sigma dx^2 - (\Sigma dx)^2}$$

(ii) Regression Equation of X on Y

$$X - \overline{X} = b_{xy}(Y - \overline{Y})$$

$$b_{xy} = \frac{N\Sigma dxdy - (\Sigma dx)(\Sigma dy)}{N\Sigma dy^2 - (\Sigma dy)^2}$$

where $dx=X-A$, $dy=Y-B$, A and B are assumed means or arbitrary values are taken from X and Y respectively.

Properties of Regression Coefficients

(i) Correlation Coefficient is the geometric mean between the regression coefficients $r = \pm\sqrt{b_{xy} \times b_{yx}}$

(ii) If one of the regression coefficients is greater than unity, the other must be less than unity.

(iii) Both the regression coefficients are of same sign.

Example 9.9

Calculate the regression coefficient and obtain the lines of regression for the following data

X	1	2	3	4	5	6	7
Y	9	8	10	12	11	13	14

Solution:

X	Y	X^2	Y^2	X^Y
1	9	1	81	9
2	8	4	64	16
3	10	9	100	30
4	12	16	144	48
5	11	25	121	55
6	13	36	169	78
7	14	49	196	98
$\sum X = 28$	$\sum Y = 77$	$\sum X^2 = 140$	$\sum Y^2 = 875$	$\sum XY = 334$

Table 9.7

$$\overline{X} = \frac{\Sigma X}{N} = \frac{28}{7} = 4\,,$$

$$\overline{Y} = \frac{\Sigma Y}{N} = \frac{77}{7} = 11$$

Regression coefficient of *X* on *Y*

$$b_{xy} = \frac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{N\,\Sigma Y^2 - (\Sigma Y)^2}$$

$$= \frac{7(334) - (28)(77)}{7(875) - (77)^2}$$

$$= \frac{2338 - 2156}{6125 - 5929}$$

$$= \frac{182}{196}$$

$$\therefore \quad b_{xy} = 0.929$$

(i) Regression equation of X on Y

$$X-\overline{X} = b_{xy}(Y-\overline{Y})$$

$$X-4 = 0.929(Y-11)$$

$$X-4 = 0.929Y-10.219$$

$\therefore$ The regression equation X on Y is $X= 0.929Y-6.219$

(ii) Regression coefficient *of* Y on X

$$b_{yx} = \frac{N\Sigma XY-(\Sigma X)(\Sigma Y)}{N\ \Sigma X^2-(\Sigma X)^2}$$

$$= \frac{7(334)-(28)(77)}{7(140)-(28)^2}$$

$$= \frac{2338-2156}{980-784}$$

$$= \frac{182}{196}$$

$$\therefore \quad b_{yx} = 0.929$$

(iii) Regression equation of Y on X

$$Y-\overline{Y} = b_{yx}(X-\overline{X})$$

$$Y-11 = 0.929\ (X-4)$$

$$Y = 0.929X-3.716+11$$

$$Y = 0.929X+7.284$$

$\therefore$ The regression equation of Y on X is $Y= 0.929X + 7.284$

Example 9.10

Calculate the two regression equations of X on Y and Y on X from the data given below, taking deviations from a actual means of X and Y.

Price(Rs.)	10	12	13	12	16	15
Amount demanded	40	38	43	45	37	43

Estimate the likely demand when the price is Rs.20.

Solution:

Calculation of Regression equation

X	$x=(X-13)$	x^2	Y	$y=(Y-41)$	y^2	xy
10	−3	9	40	−1	1	3
12	−1	1	38	−3	9	3
13	0	0	43	2	4	0
12	−1	1	45	4	16	−4
16	3	9	37	−4	16	−12
15	2	4	43	2	4	4
$\sum X=78$	$\sum x=0$	$\sum x^2=24$	$\sum Y=246$	$\sum y=0$	$\sum y^2=50$	$\sum xy=-6$

Table 9.8

(i) Regression equation of X on Y

$$X-\overline{X} = r\frac{\sigma_x}{\sigma_y}(Y-\overline{Y})$$

$$\overline{X} = \frac{78}{6}=13\,,\ \overline{Y} = \frac{246}{6}=41$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}=\frac{\Sigma xy}{\Sigma y^2} = \frac{-6}{50}=-0.12$$

$$X-13 = -0.12\,(Y-41)$$

$$X-13 = -0.12Y+4.92$$

$$X = -0.12Y+17.92$$

(ii) Regression Equation of Y on X

$$Y-\overline{Y} = r\frac{\sigma_y}{\sigma_x}(X-\overline{X})$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}=\frac{\Sigma xy}{\Sigma x^2} = -\frac{6}{24}=-0.25$$

$$Y-41 = -0.25\,(X-13)$$

$$Y-41 = -0.25X+3.25$$

$$Y = -0.25X+44.25$$

When X is 20, Y will be

$$Y = -0.25\ (20)+44.25$$

$$= -5+44.25$$

$$= 39.25 \text{ (when the price is Rs. 20, the likely demand is 39.25)}$$

Example 9.11

Obtain regression equation of Y on X and estimate Y when X=55 from the following

X	40	50	38	60	65	50	35
Y	38	60	55	70	60	48	30

Solution:

X	Y	$dx=(X-48)$	dx^2	$dy=(Y-50)$	dy^2	$dxdy$
40	38	–8	64	–12	144	96
50	60	2	4	10	100	20
38	55	–10	100	5	25	–50
60	70	12	144	20	400	240
65	60	17	289	10	100	170
50	48	2	4	-2	4	-4
35	30	-13	169	-20	400	260
$\sum X = 338$	$\sum Y = 361$	$\sum dx = 2$	$\sum dx^2 = 774$	$\sum dy = 11$	$\sum dy^2 = 1173$	$\sum dxdy = 732$

Table 9.9

$$\overline{X} = \frac{\Sigma X}{N} = \frac{338}{7} = 48.29$$

$$\overline{Y} = \frac{\Sigma Y}{N} = \frac{361}{7} = 51.57$$

(i) Regression coefficients of Y on X

$$b_{yx} = \frac{N\Sigma dx\, dy - (\Sigma dx)(\Sigma dy)}{N\ \Sigma dx^2 - (\Sigma dx)^2}$$

$$= \frac{7(732)-(2)(11)}{7(774)-(2)^2}$$

$$= \frac{5124-22}{5418-4}$$

$$= \frac{5102}{5414}$$

$$= 0.942$$

$$b_{yx} = 0.942$$

(ii) Regression equation of Y on X

$$Y - \overline{Y} = b_{yx}(X - \overline{X})$$

$$Y - 51.57 = 0.942(X - 48.29)$$

$$Y = 0.942X - 45.49 + 51.57 = 0.942 \times -45.49 + 51.57$$

$$Y = 0.942X + 6.08$$

$\therefore$ The regression equation of Y on X is $Y = 0.942X + 6.08$

Estimation of Y when $X = 55$

$Y = 0.942(55) + 6.08 = 57.89$

Example 9.12

Find the means of X and Y variables and the coefficient of correlation between them from the following two regression equations:

$2Y - X - 50 = 0$

$3Y - 2X - 10 = 0.$

Solution:

We are given

$$2Y - X - 50 = 0 \qquad \dots (1)$$

$$3Y - 2X - 10 = 0 \qquad \dots (2)$$

Solving equation (1) and (2)

We get $Y = 90$

Putting the value of Y in equation (1)

We get $X = 130$

Hence $\overline{X} = 130$ and $\overline{Y} = 90$

Calculating correlation coefficient

Let us assume equation (1) be the regression equation of Y on X

$$2Y = X+50$$

$$Y = \frac{1}{2}X+25 \text{ therefore } b_{yx} = \frac{1}{2}$$

Clearly equation (2) would be treated as regression equation of X on Y

$$3Y-2X-10 = 0$$

$$2X = 3Y-10$$

$$X = \frac{3}{2}Y-5 \text{ therefore } b_{xy} = \frac{3}{2}$$

The Correlation coefficient $r = \pm\sqrt{b_{xy} \times b_{yx}}$

$$r = \sqrt{\frac{1}{2} \times \frac{3}{2}} = 0.866$$

NOTE

It may be noted that in the above problem one of the regression coefficient is greater than 1 and the other is less than 1. Therefore our assumption on given equations are correct.

Example 9.13

Find the means of X and Y variables and the coefficient of correlation between them from the following two regression equations:

$$4X-5Y+33 = 0$$

$$20X-9Y-107 = 0$$

Solution:

We are given

$$4X-5Y+33 = 0 \quad \dots (1)$$

$$20X-9Y-107 = 0 \quad \dots (2)$$

Solving equation (1) and (2)

We get $\quad Y = 17$

Putting the value of Y in equation (1)

We get $X = 13$

Hence $\overline{X} = 13$ and $\overline{Y} = 17$

Calculating correlation coefficient

Let us assume equation (1) be the regression equation of X on Y

$$4X = 5Y - 33$$

$$X = \frac{1}{4}(5Y - 33)$$

$$X = \frac{5}{4}Y - \frac{33}{4}$$

$$b_{xy} = \frac{5}{4} = 1.25$$

Let us assume equation (2) be the regression equation of Y on X

$$9Y = 20X - 107$$

$$Y = \frac{1}{9}(20X - 107)$$

$$Y = \frac{20}{9}X - \frac{107}{9}$$

$$b_{yx} = \frac{20}{9} = 2.22$$

But this is not possible because both the regression coefficient are greater than 1. So our above assumption is wrong. Therefore treating equation (1) has regression equation of Y on X and equation (2) has regression equation of X on Y. So we get

$$b_{yx} = \frac{4}{5} = 0.8$$

and $$b_{xy} = \frac{9}{20} = 0.45$$

The Correlation coefficient

$$r = \pm\sqrt{b_{xy} \times b_{xy}}$$

$$r = \sqrt{0.45 \times 0.8} = 0.6$$

Example 9.14

The following table shows the sales and advertisement expenditure of a form

	Sales	Advertisement expenditure (Rs. Crores)
Mean	40	6
SD	10	1.5

Coefficient of correlation $r= 0.9$. Estimate the likely sales for a proposed advertisement expenditure of Rs. 10 crores.

Solution:

Let the sales be X and advertisement expenditure be Y

Given $\overline{X}=40, \overline{Y}=6, \sigma_x=10, \sigma_y=1.5$ and $r=0.9$

Equation of line of regression x on y is

$$
\begin{aligned}
X-\overline{X} &= r\frac{\sigma_x}{\sigma_y}(Y-\overline{Y}) \\
X-40 &= (0.9)\frac{10}{1.5}(Y-6) \\
X-40 &= 6Y-36 \\
X &= 6Y+4
\end{aligned}
$$

When advertisement expenditure is 10 crores i.e., $Y=10$ then sales $X=6(10)+4=64$ which implies sales is 64.

Example 9.15

There are two series of index numbers P for price index and S for stock of the commodity. The mean and standard deviation of P are 100 and 8 and of S are 103 and 4 respectively. The correlation coefficient between the two series is 0.4. With these data obtain the regression lines of P on S and S on P.

Solution:

Let us consider X for price P and Y for stock S. Then the mean and SD for P is considered as $\overline{X}=100$ and $\sigma_x=8$. respectively and the mean and SD of S is considered as $\overline{Y}=103$ and $\sigma_y=4$. The correlation coefficient between the series is $r(X,Y)=0.4$

Let the regression line X on Y be

$$
\begin{aligned}
X-\overline{X} &= r\frac{\sigma_x}{\sigma_y}(Y-\overline{Y}) \\
X-100 &= (0.4)\frac{8}{4}(Y-103) \\
X-100 &= 0.8(Y-103) \\
X-0.8Y-17.6 &= 0 \quad \text{or} \quad X=0.8Y+17.6
\end{aligned}
$$

The regression line Y on X be $Y-\overline{Y}=r\,\frac{\sigma_y}{\sigma_x}(X-\overline{X})$

$$Y-103 = (0.4)\,\frac{4}{8}\,(X-100)$$

$$Y-103 = 0.2\,(X-100)$$

$$Y-103 = 0.2\,X-20$$

$$Y = 0.2\,X+83 \quad \text{or} \quad 0.2\,X-Y+83=0$$

Example 9.16

For 5 pairs of observations the following results are obtained $\Sigma X=15$, $\Sigma Y=25$, $\Sigma X^2=55$, $\Sigma Y^2=135$, $\Sigma XY=83$ Find the equation of the lines of regression and estimate the value of X on the first line when $Y=12$ and value of Y on the second line if $X=8$.

Solution:

Here $N=5$, $\overline{X}=\frac{\Sigma X}{N}=\frac{15}{5}=3$, $\overline{Y}=\frac{\Sigma Y}{N}=\frac{25}{5}=5$ and the regression coefficient

$$b_{xy}=\frac{N\Sigma XY-\Sigma X\Sigma Y}{N\Sigma Y^2-(\Sigma Y)^2}=\frac{5(83)-(15)(25)}{5(135)-(25)^2}=0.8$$

The regression line of X on Y is

$$X-\overline{X} = b_{xy}\,(Y-\overline{Y})$$

$$X-3 = 0.8(Y-5)$$

$$X = 0.8\,Y-1$$

When $Y=12$, the value of X is estimated as

$$X = 0.8\,(12)-1=8.6$$

The regression coefficient

$$b_{yx} = \frac{N\Sigma XY-\Sigma X\Sigma Y}{N\Sigma X^2-(\Sigma X)^2}$$

$$= \frac{5(83)-(15)(25)}{5(55)-(15)^2}=0.8$$

Thus $b_{yx}=0.8$ then the regression line Y on X is

$$Y-\overline{Y} = b_{yx}\,(X-\overline{X})$$

$$Y-5 = 0.8(X-3)$$

$$Y = 0.8X+2.6$$

When X=8 the value of Y is estimated as

$$Y = 0.8(8)+2.6$$

$$Y = 9$$

Example 9.17

The two regression lines are $3X+2Y=26$ and $6X+3Y=31$. Find the correlation coefficient.

Solution:

Let the regression equation of Y on X be

$$3X+2Y = 26$$

$$2Y = -3X+26$$

$$Y = \frac{1}{2}(-3X+26)$$

$$Y = -1.5X+13$$

$$r\frac{\sigma_y}{\sigma_x} = -1.5$$

Implies $\quad b_{yx} = r\frac{\sigma_y}{\sigma_x} = -1.5$

Let the regression equation of X on Y be

$$6X+3Y = 31$$

$$X = \frac{1}{6}(-3Y+31) = -0.5Y+5.17$$

$$r\frac{\sigma_x}{\sigma_y} = -0.5$$

Implies $\quad b_{xy} = r\frac{\sigma_x}{\sigma_y} = -0.5$

$$r = \pm\sqrt{b_{xy}.b_{yx}}$$

$= -\sqrt{(-1.5).(-0.5)}$ (Since both the regression coefficient are negative r is negative)

$\therefore \quad r = -0.866$

Example 9.18

In a laboratory experiment on correlation research study the equation of the two regression lines were found to be $2X-Y+1=0$ and $3X-2Y+7=0$. Find the means of X and Y. Also work out the values of the regression coefficient and correlation between the two variables X and Y.

Solution:

Solving the two regression equations we get mean values of X and Y

$$2X-Y = -1 \qquad \text{... (1)}$$

$$3X-2Y = -7 \qquad \text{... (2)}$$

Solving equation (1) and equation (2) We get $X=5$ and $Y=11$

Therefore the regression line passing through the means $\overline{X}=5$ and $\overline{Y}=11$

The regression equation of Y on X is $3X-2Y=-7$

$$2Y = 3X+7$$

$$Y = \frac{1}{2}(3X+7)$$

$$Y = \frac{3}{2}X+\frac{7}{2}$$

$$\therefore \quad b_{yx} = \frac{3}{2}\ (>1)$$

The regression equation of X on Y is

$$2X-Y = -1$$

$$2X = Y-1$$

$$X = \frac{1}{2}(Y-1)$$

$$X = \frac{1}{2}Y-\frac{1}{2}$$

$$\therefore \quad b_{xy} = \frac{1}{2}$$

The regression coefficients are positive

$$r = \pm\sqrt{b_{xy}.b_{yx}} = \pm\sqrt{\frac{3}{2}\times\frac{1}{2}}$$

$$= \sqrt{\frac{3}{2}\times\frac{1}{2}}$$

$$= \sqrt{\frac{3}{4}}$$

$$= 0.866$$

$$\therefore \quad r = 0.866$$

Example 9.19

For the given lines of regression $3X-2Y=5$and $X-4Y=7$. Find

(i) Regression coefficients

(ii) Coefficient of correlation

Solution:

(i) First convert the given equations Y on X and X on Y in standard form and find their regression coefficients respectively.

Given regression lines are

$$3X-2Y = 5 \quad \dots (1)$$

$$X-4Y = 7 \quad \dots (2)$$

Let the line of regression of X on Y is

$$3X-2Y = 5$$

$$3X = 2Y+5$$

$$X = \frac{1}{3}(2Y+5)$$

$$X = \frac{1}{3}(2Y+5)$$

$$X = \frac{2}{3}Y+\frac{5}{3}$$

$\therefore$ Regression coefficient of X on Y is

$$b_{xy} = \frac{2}{3}(<1)$$

Let the line of regression of Y on X is

$$X-4Y = 7$$

$$-4Y = -X+7$$

$$4Y = X-7$$

$$Y = \frac{1}{4}(X-7)$$

$$Y = \frac{1}{4}X - \frac{7}{4}$$

$\therefore$ Regression coefficient of Y on X is

$$b_{yx} = \frac{1}{4}(<1)$$

Coefficient of correlation

Since the two regression coefficients are positive then the correlation coefficient is also positive and it is given by

$$r = \sqrt{b_{yx} \cdot b_{xy}}$$

$$= \sqrt{\frac{2}{3} \cdot \frac{1}{4}}$$

$$= \sqrt{\frac{1}{6}}$$

$$= 0.4082$$

$$\therefore \quad r = 0.4082$$

Exercise 9.2

1. From the data given below

Marks in Economics:	25	28	35	32	31	36	29	38	34	32
Marks in Statistics:	43	46	49	41	36	32	31	30	33	39

Find (a) The two regression equations, (b) The coefficient of correlation between marks in Economics and statistics, (c) The mostly likely marks in Statistics when the marks in Economics is 30.

2. The heights (in cm.) of a group of fathers and sons are given below

Heights of fathers:	158	166	163	165	167	170	167	172	177	181
Heights of Sons :	163	158	167	170	160	180	170	175	172	175

Find the lines of regression and estimate the height of son when the height of the father is 164 cm.

3. The following data give the height in inches (X) and the weight in lb. (Y) of a random sample of 10 students from a large group of students of age 17 years:

X	61	68	68	64	65	70	63	62	64	67
Y	112	123	130	115	110	125	100	113	116	125

Estimate weight of the student of a height 69 inches.

4. Obtain the two regression lines from the following data N=20, ΣX=80, ΣY=40, ΣX^2=1680, ΣY^2=320 and ΣXY=480

5. Given the following data, what will be the possible yield when the rainfall is 29``

	Rainfall	Production
Mean	25``	40 units per acre
Standard Deviation	3``	6 units per acre

Coefficient of correlation between rainfall and production is 0.8

6. The following data relate to advertisement expenditure(in lakh of rupees) and their corresponding sales(in crores of rupees)

Advertisement expenditure	40	50	38	60	65	50	35
Sales	38	60	55	70	60	48	30

Estimate the sales corresponding to advertising expenditure of Rs. 30 lakh.

7. You are given the following data:

	X	Y
Arithmetic Mean	36	85
Standard Deviation	11	8

If the Correlation coefficient between X and Y is 0.66, then find (i) the two regression coefficients, (ii) the most likely value of Y when X=10

8. Find the equation of the regression line of Y on X, if the observations (X_i, Y_i) are the following (1,4) (2,8) (3,2) (4,12) (5, 10) (6, 14) (7, 16) (8, 6) (9, 18)

9. A survey was conducted to study the relationship between expenditure on accommodation (X) and expenditure on Food and Entertainment (Y) and the following results were obtained:

	Mean	SD
Expenditure on Accommodation	Rs. 178	63.15
Expenditure on Food and Entertainment	Rs 47.8	22.98
Coefficient of Correlation	0.43	

Write down the regression equation and estimate the expenditure on Food and Entertainment, if the expenditure on accommodation is Rs. 200.

10. For 5 observations of pairs of (X, Y) of variables X and Y the following results are obtained. $\Sigma X=15$, $\Sigma Y=25$, $\Sigma X^2=55$, $\Sigma Y^2=135$, $\Sigma XY=83$. Find the equation of the lines of regression and estimate the values of X and Y if $Y=8$; $X=12$.

11. The two regression lines were found to be $4X-5Y+33=0$ and $20X-9Y-107=0$. Find the mean values and coefficient of correlation between X and Y.

12. The equations of two lines of regression obtained in a correlation analysis are the following $2X=8-3Y$ and $2Y=5-X$. Obtain the value of the regression coefficients and correlation coefficient.

Exercise 9.3

IL8BC4

Choose the correct answers

1. Example for positive correlation is

   (a) Income and expenditure (b) Price and demand

   (c) Repayment period and EMI (d) Weight and Income

2. If the values of two variables move in same direction then the correlation is said to be

   (a) Negative (b) positive

   (c) Perfect positive (d) No correlation

3. If the values of two variables move in opposite direction then the correlation is said to be

   (a) Negative (b) Positive (c) Perfect positive (d) No correlation

4. Correlation co-efficient lies between

(a) 0 to ∞ (b) -1 to $+1$ (c) -1 to 0 (d) -1 to ∞

5. If $r(X,Y) = 0$ the variables X and Y are said to be

(a) Positive correlation (b) Negative correlation

(c) No correlation (d) Perfect positive correlation

6. The correlation coefficient from the following data N=25, ΣX=125, ΣY=100, ΣX^2=650, ΣY^2=436, ΣXY=520

(a) 0.667 (b) -0.006 (c) -0.667 (d) 0.70

7. From the following data, N=11, ΣX=117, ΣY=260, ΣX^2=1313, ΣY^2=6580, ΣXY=2827the correlation coefficient is

(a) 0.3566 (b) -0.3566 (c) 0 (d) 0.4566

8. The correlation coefficient is

(a) $r(X,Y)=\dfrac{\sigma_x\sigma_y}{\text{cov}(x,y)}$ (b) $r(X,Y)=\dfrac{\text{cov}(x,y)}{\sigma_x\sigma_y}$

(c) $r(X,Y)=\dfrac{\text{cov}(x,y)}{\sigma_y}$ (d) $r(X,Y)=\dfrac{\text{cov}(x,y)}{\sigma_x}$

9. The variable whose value is influenced or is to be predicted is called

(a) dependent variable (b) independent variable
(c) regressor (d) explanatory variable

10. The variable which influences the values or is used for prediction is called

(a) Dependent variable (b) Independent variable

(c) Explained variable (d) Regressed

11. The correlation coefficient

(a) $r=\pm\sqrt{b_{xy}\times b_{yx}}$ (b) $r=\dfrac{1}{b_{xy}\times b_{yx}}$

(c) $r=b_{xy}\times b_{yx}$ (d) $r=\pm\sqrt{\dfrac{1}{b_{xy}\times b_{yx}}}$

12. The regression coefficient of X on Y

(a) $b_{xy}=\dfrac{N\Sigma dx\,dy-(\Sigma dx)(\Sigma dy)}{N\,\Sigma dy^2-(\Sigma dy)^2}$ (b) $b_{yx}=\dfrac{N\Sigma\,dx\,dy-(\Sigma dx)(\Sigma dy)}{N\,\Sigma dy^2-(\Sigma dy)^2}$

(c) $b_{xy}=\dfrac{N\Sigma dxdy-(\Sigma dx)(\Sigma dy)}{N\,\Sigma dx^2-(\Sigma dx)^2}$ (d) $b_{xy}=\dfrac{N\Sigma xy-(\Sigma x)(\Sigma y)}{\sqrt{N\Sigma x^2-(\Sigma x)^2}\times\sqrt{N\Sigma y^2-(\Sigma y)^2}}$

13. The regression coefficient of Y on X

(a) $b_{xy} = \dfrac{N\Sigma dx\,dy - (\Sigma dx)(\Sigma dy)}{N\,\Sigma dy^2 - (\Sigma dy)^2}$

(b) $b_{yx} = \dfrac{N\Sigma dxdy - (\Sigma dx)(\Sigma dy)}{N\Sigma dy^2 - (\Sigma dy)^2}$

(c) $b_{yx} = \dfrac{N\Sigma dx\,dy - (\Sigma dx)(\Sigma dy)}{N\Sigma dx^2 - (\Sigma dx)^2}$

(d) $b_{xy} = \dfrac{N\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{N\Sigma x^2 - (\Sigma x)^2} \times \sqrt{N\Sigma y^2 - (\Sigma y)^2}}$

14. When one regression coefficient is negative, the other would be

(a) Negative (b) Positive (c) Zero (d) None of them

15. If X and Y are two variates, there can be atmost

(a) One regression line (b) two regression lines

(c) three regression lines (d) more regression lines

16. The lines of regression of X on Y estimates

(a) X for a given value of Y (b) Y for a given value of X

(c) X from Y and Y from X (d) none of these

17. Scatter diagram of the variate values (X,Y) give the idea about

(a) functional relationship (b) regression model

(c) distribution of errors (d) no relation

18. If regression co-efficient of Y on X is 2, then the regression co-efficient of X on Y is

(a) $\leq \frac{1}{2}$ (b) 2 (c) $> \frac{1}{2}$ (d) 1

19. If two variables moves in decreasing direction then the correlation is

(a) positive (b) negative (c) perfect negative (d) no correlation

20. The person suggested a mathematical method for measuring the magnitude of linear relationship between two variables say X and Y is

(a) Karl Pearson (b) Spearman

(c) Croxton and Cowden (d) Ya Lun Chou

21. The lines of regression intersect at the point

(a) (X,Y) (b) $(\overline{X},\overline{Y})$ (c) $(0,0)$ (d) (σ_x,σ_y)

22. The term regression was introduced by

(a) R.A Fisher (b) Sir Francis Galton

(c) Karl Pearson (d) Croxton and Cowden

23. If $r=-1$, then correlation between the variables

(a) perfect positive (b) perfect negative

(c) negative (d) no correlation

24. The coefficient of correlation describes

(a) the magnitude and direction (b) only magnitude

(c) only direction (d) no magnitude and no direction.

25. If $\text{Cov}(x,y)=-16.5$, $\sigma_x^2=2.89$, $\sigma_y^2=100$. Find correlation coefficient.

(a) -0.12 (b) 0.001 (c) -1 (d) -0.97

Miscellaneous Problems

1. Find the coefficient of correlation for the following data:

X	35	40	60	79	83	95
Y	17	28	30	32	38	49

2. Calculate the coefficient of correlation from the following data:

$$\Sigma X=50,\ \Sigma Y=-30,\ \Sigma X^2=290,\ \Sigma Y^2=300,\ \Sigma XY=-115,\ N=10$$

3. Calculate the correlation coefficient from the data given below:

X	1	2	3	4	5	6	7	8	9
Y	9	8	10	12	11	13	14	16	15

4. Calculate the correlation coefficient from the following data:

$\sum X=125$, $\sum Y=100$, $\sum X^2=650$, $\sum Y^2=436$, $\sum XY=520$, $N=25$

5. A random sample of recent repair jobs was selected and estimated cost , actual cost were recorded.

Estimated cost	30	45	80	25	50	97	47	40
Actual cost	27	48	73	29	63	87	39	45

Calculate the value of spearman's correlation.

6. The following data pertains to the marks in subjects A and B in a certain examination. Mean marks in A = 39.5, Mean marks in B= 47.5 standard deviation of marks in A =10.8 and Standard deviation of marks in B= 16.8. coefficient of correlation between marks in A and marks in B is 0.42. Give the estimate of marks in B for candidate who secured 52 marks in A.

7. X and Y are a pair of correlated variables. Ten observations of their values (X,Y) have the following results. $\sum X=55$, $\sum XY=350$, $\sum X^2=385$, $\sum Y=55$, Predict the value of y when the value of X is 6.

8. Find the line regression of Y on X

X	1	2	3	4	5	8	10
Y	9	8	10	12	14	16	15

9. Using the following information you are requested to (i) obtain the linear regression of Y on X (ii) Estimate the level of defective parts delivered when inspection expenditure amounts to Rs.28,000 $\sum X=424$, $\sum Y=363$, $\sum X^2=21926$, $\sum Y^2=15123$, $\sum XY=12815$, $N=10$. Here X is the expenditure on inspection, Y is the defective parts delivered

10. The following information is given.

	X(in Rs.)	**Y (in Rs.)**
Arithmetic Mean	6	8
Standard Deviation	5	$\frac{40}{3}$

Coefficient of correlation between X and Y is $\frac{8}{15}$. Find (i) The regression Coefficient of Y on X (ii) The most likely value of Y when X =Rs.100

CASE STUDY-1

Mr. Bean visited a departmental store in Triplicane at Chennai on 1st March 2018 and chooses 15 different types of food items that include nutrition information on its packaging. For each food Mr. Bean observed and identified the amount of fat (gms) and the sodium content(mgs/100gms) per serving and recorded in the following table.

S.No	Product Items	Fat(gm/100gms)	Sodium(mg / 100gms)
1	Dates	0.4	74.4
2	Appalam	0.26	1440
3	Energy drink	1.8	136
4	Gulabjamun Powder	10.4	710
5	Atta	2.2	4.97
6	Athi fruits	0.14	2
7	Alu Muttar mix	5	440
8	Popcorn	2.32	51.38
9	Perungayum	0.37	40
10	Mushroom	31	11.73
11	Friut juice	0.1	74
12	Choclate	0.8	0.09
13	Rava	9	575
14	Biscuit	19.7	498
15	Snacks	33.5	821

Mr. Bean wants to establish some statistical relationship between the above mentioned food contents. Here the variable under study are X and Y which represents the amount of Fat content and the amount of Sodium content in each food items respectively. Thus Mr. Bean gets a pair of values (X,Y) for each food item. Mr. Bean further found the average fat content in all the 15 food items is $\overline{X}$ =7.8 (gms/100g) and the average sodium content in all the items is $\overline{Y}$ =325.23(mg/100g). Further, it was identified that the minimum amount of fat contained in Tropicana fruit juice is 0.1(g/100gms) and the maximum amount of fat contained in Lays is 33.5(g/100gms). Thus the fat contained in all the 15 food items is ranging from 0.1(g/100gms) to 33.5(g/100gms). Similarly, the minimum amount of sodium content contained in Kelloggys Choco is 0.09(mg/100gm) and the maximum amount of sodium content in Bhindhu appalam is 1440(mg/100gm). So, the measure of range of fat content and sodium content in all the 15 items are 33.4gm and 1439.91(mg/100gm) respectively. Besides, Mr. Bean is interested in knowing the variation of each individual item from the mean of all observations. He attempted another measure of dispersion known as standard deviation. The measure of standard deviation indicates that there is an average deviation of 11.3 (g/100gms) in fat content and 420.14(mg/100gms) in sodium content of all the 15 food items. Further, Mr. Bean is keen on finding the association between the variables X and Y. So, the correlation analysis has been carried out. The correlation coefficient $r(X,Y)$=0.2285 indicates that there is 22.86% positive association between sodium content and the amount of fat content. Thus from this study Mr. Bean inferred and convinced that the nutrition information on the packaging of each food item is sufficient.

CASE STUDY-2

We collected data relating to the gold price (per gram) in two places namely Chennai Market and Mumbai Market for ten days from 20th Feb 2018 to 1st March 2018 and the same is recorded below.

Date	20th Feb	21st Feb	22-Feb	23-Feb	24-Feb	25-Feb	26-Feb	27-Feb	28-Feb	1st Mar
Chennai X	2927	2912	2919	2912	2921	2921	2927	2924	2908	2893
Mumbai Y	2923	2910	2907	2920	2919	2919	2925	2918	2902	2895

Do we agree that the price of gold in Chennai market has its impact on Mumbai market?

Let X denotes the gold price per gm in Chennai market and Y denotes the gold price per gm in Mumbai market. The actual observations given in the table indicates that the gold price ranges from Rs. 2893 to 2927 in Chennai market and the gold price range from Rs 2895(per gram) to Rs. 2925(per gram) in Mumbai market. Further it is to be noted that there is some oscillations in the gold price rate dated between 20th Feb to 24th Feb and 25th Feb price is remain same as the previous day that is Feb 24th. It may be due to holiday of Gold markets. It is clear from the above table the price of gold rate go on rapidly decreasing from 27th Feb to 1st March. The same fluctuations is observed in Mumbai market from Feb 20th to 24th and remain same on 24th and 25th and rapidly decreasing from Feb 27th to 1st March . So, the market trend in respect of gold rate is same in two markets. We found that the average gold price in Chennai market during these 10 days is Rs. 2916.4 (per gm) and the Mumbai market is Rs.2913.8(per gm). The variation among the prices of gold in 10 days. To identify the variation of each individual observation from the mean of all observations. We make use of the best measure known as standard deviation. In this study it is found that the price of gold has a average deviation of Rs. 10 (approximately) in Chennai market and Rs. 9(approximately) in Mumbai market. To verify the consistency of the prices between the two cities, we attempt coefficient of variation which expresses the percentage of variation. Coefficient of variation of gold price in Chennai market is

$$CV_X = \frac{\sigma_X}{X} \times 100 = \frac{10}{2916.4} \times 100 = 0.343\%$$

Similarly, Coefficient of variation of gold price in Mumbai market is

$$CV_Y = \frac{\sigma_Y}{Y} \times 100 = \frac{9}{2913.8} \times 100 = 0.31\%$$

Comparison of these coefficients of variations we inferred that the Mumbai market has consistent or more stable in price of gold.

Further to examine the linear relationship between the two variables X and Y. We carry out correlation analysis results $r(X,Y) = 0.8682$. It indicates that there is a positive correlation in price of gold between Mumbai market and Chennai market. Do you think finding a regression line makes sense here?

Summary

- The term correlation refers to the degree of relationship between two or more variables.
- Scatter diagram is a graphic device for finding correlation between two variables.
- Karl Pearson correlation coefficient $r(x,y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$
- Correlation coefficient r lies between -1 and 1. (i.e) $-1 \le r \le 1$
- When $r=+1$, then the correlation is perfect positive
- When $r=-1$, then the correlation is perfect negative
- When $r=0$, then there is no relationship between the variables, (i.e) the variables are uncorrelated.
- Rank correlation deals with qualitative characteristics.

- Spearman's rank correlation coefficient formula ρ is given by

$$\rho = 1 - \frac{6\Sigma d^2}{N(N^2-1)}$$

where d = The difference between two ranks = $R_X - R_Y$

- N = Number of paired observations.

- Correlation represents linear relationship between the variables but the regression helps to estimate (or predict) one variable by using the other variable.

- Regression lines of

(i) X on Y is $X - \overline{X} = b_{xy}(Y - \overline{Y})$ (ii) Y on X is $Y - \overline{Y} = b_{yx}(X - \overline{X})$

- The two regression lines passing through their respective means of X and Y

- Calculation of the regression coefficients.

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} \text{ and } b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

- The Properties of regression coefficients.

(i) $r = \sqrt{b_{yx} \times b_{xy}}$

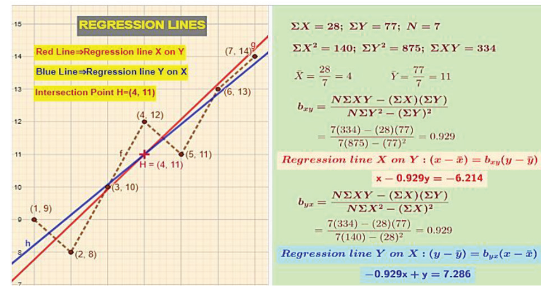
(ii) both the regression coefficients cannot be greater than one.

(iii) Both the regression coefficients have same sign.

GLOSSARY	
Abnormal	அசாதாரனமான
Approximate	தோராயமாக
Assumed Mean	ஊகிக்கப்பட்ட சராசரி
Bivariate analysis	இருமாறி பகுப்பாய்வு
Characteristics	பண்புகள்
Closeness	பொருத்தமுடைய
Corrleation	ஒட்டுறவு
Deviations	விலக்கம்
fluctuate	ஏற்ற இறக்கம்
Interpretation	விளக்கம்
Negative Correlation	எதிர்மறை ஒட்டுறவு
Positive Correlation	நேரிடை ஒட்டுறவு
Random variables	சமவாய்ப்பு மாறிகள்
Regression analysis	தொடர்பு போக்கு ஆய்வு
Relative Variable	சார்ந்த மாறி
Univariate analysis	ஒருமாறி பகுப்பாய்வு

ICT Corner

Correlation and regression analysis



Expected Outcome ⇒

Step – 1

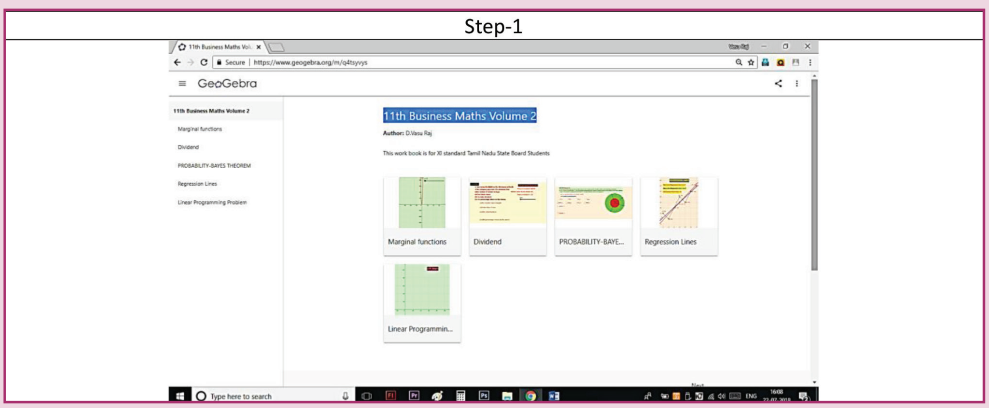
Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "11th Business Maths Volume-2" will appear. In that there are several worksheets related to your Text Book.

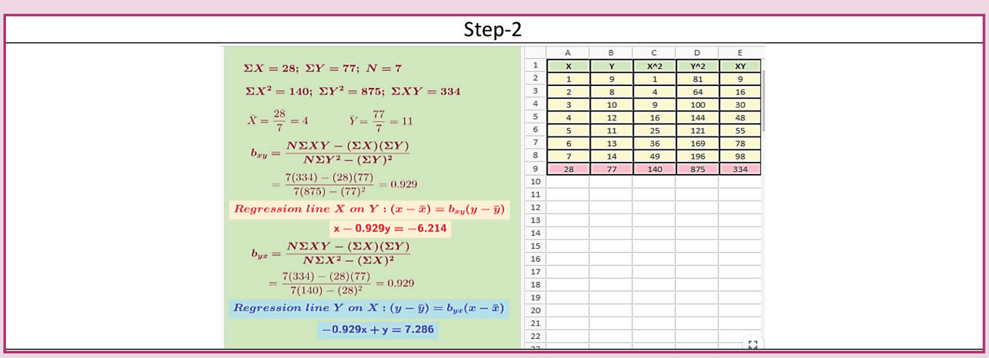
Step-2

Select the work sheet "Regression lines" work out the problem for the data given and workout as given and verify the steps. See the graph of regression line x on y and regression line y on x and check the intersection of these two lines.(Mean of x, mean of y) you can change the data x and Y in the spread sheet for new problem.

Step 1



Step 2



Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

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Chapter 10

OPERATIONS RESEARCH

Learning Objectives

After studying this chapter, the students will be able to understand

- Formulation of Linear Programming Problem
- Solution of LPP by graphical method.
- Construction of network of a project
- Project completion time by Critical Path Method(CPM)

ILH7DR

Introduction

L.V. Kantorovich

During the world war II, the military management in England recruited a team of scientists, engineers and mathematicians to study the strategic and tactical problems of air and land defence. Their objective was to determine the best utilization of limited military resources like ammunition, food and other things needed for war. This group of scientists formed the first operations research team. The name operations research was apparently coined because the team was dealing with the research on (military) operations. Operations research team helped in developing strategies for mining operations, inventing new flight pattern and planning of sea mines. Following the end of war the success of military team attracted the attention of industrial managers who were seeking solutions to their complex type of problems.

It is not possible to give uniformly acceptable definition of operations research. The following is the definition of operations research published on behalf of UK operational research society. "operations research is the application of scientific methods to complex problems arising from operations involving large system of men, machines, materials and money in industry, business, government and defence."

An operations research model is defined as any abstract or idealized representation of real life system or situation. The objective of the model is to identify significant factors and inter-relationship. Here we study only two models namely linear programming problem and network analysis.

10.1. Linear programming problem

The Russian Mathematician L.V. Kantorovich applied mathematical model to solve linear programming problems. He pointed out in 1939 that many classes of problems which arise in production can be defined mathematically and therefore can be solved numerically. This decision making technique was further developed by George B. Dantziz. He formulated the general linear programming problem and developed simplex method (1947) to solve complex real time applications. Linear programming is one of the best optimization technique from theory, application and computation point of view.

Definition:

Linear Programming Problem(LPP) is a mathematical technique which is used to optimize (maximize or minimize) the objective function with the limited resources.

Mathematically, the general linear programming problem (LPP) may be stated as follows.

Maximize or Minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the conditions (constraints)

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq (or = or \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq (or = or \geq) b_2$$

$$\ldots \quad \ldots. \quad \ldots. \quad \ldots. \quad \ldots$$

$$\ldots \quad \ldots. \quad \ldots. \quad \ldots. \quad \ldots$$

$$a_{m1}x1 + a_{m2}x_2 + ... + a_{mn}x_n (\leq or = or \geq) b_m$$

$$x_1, x_2 \ldots\ldots. x_n \geq 0$$

Short form of LPP

Maximize or Minimize $\quad Z = \sum_{j=1}^{n} c_jx_j$

Subject to $\sum_{j=1}^{n} a_{ij}x_j \leq (or = or \geq) b_i$, i = 1,2,3,…,m ... (1)

and $x_j \geq 0$ ... (2)

Some useful definitions:

Objective function:

A function $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ which is to be optimized (maximized or minimized) is called objective function.

Decision variable:

The decision variables are the variables, which has to be determined x_j, $j = 1,2,3,\ldots,n$, to optimize the objective function.

Constraints:

There are certain limitations on the use of limited resources called constraints.

$\sum_{j=1}^{n} a_{ij} x_j \leq (or = or \geq) b_i$, $i = 1,2,3,\ldots, m$ are the constraints.

Solution:

A set of values of decision variables x_j, $j=1,2,3,\ldots, n$ satisfying all the constraints of the problem is called a solution to that problem.

Feasible solution:

A set of values of the decision variables that satisfies all the constraints of the problem and non-negativity restrictions is called a feasible solution of the problem.

Optimal solution:

Any feasible solution which maximizes or minimizes the objective function is called an optimal solution.

Feasible region:

The common region determined by all the constraints including non-negative constraints $x_j \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.

DO YOU KNOW?

Linear Programming Problem helps a farmer to produce the best crop by minimising risk and maximizing profit.

10.1.1 Mathematical formulation of a linear programming problem:

The procedure for mathematical formulation of a linear programming problem consists of the following steps.

(i) Identify the decision variables.

(ii) Identify the objective function to be maximized or minimized and express it as a linear function of decision variables.

(iii) Identify the set of constraint conditions and express them as linear inequalities or equations in terms of the decision variables.

DO YOU KNOW?

Linear Programming Problem is used in determining shortest routes for travelling salesman.

Example 10.1

A furniture dealer deals only two items viz., tables and chairs. He has to invest Rs.10,000/- and a space to store atmost 60 pieces. A table cost him Rs.500/– and a chair Rs.200/–. He can sell all the items that he buys. He is getting a profit of Rs.50 per table and Rs.15 per chair. Formulate this problem as an LPP, so as to maximize the profit.

Solution:

(i) **Variables:** Let x_1 and x_2 denote the number of tables and chairs respectively.

(ii) Objective function:

$$\text{Profit on } x_1 \text{ tables} = 50\, x_1$$

$$\text{Profit on } x_2 \text{ chairs} = 15\, x_2$$

$$\text{Total profit} = 50\, x_1 + 15 x_2$$

Let $Z = 50\, x_1 + 15\, x_2$, which is the objective function.

Since the total profit is to be maximized, we have to

maximize $Z=50x_1+15x_2$

(iii) Constraints:

The dealer has a space to store atmost 60 pieces

$x_1+x_2 \leq 60$

The cost of x_1 tables = Rs.500 x_1

The cost of x_2 tables = Rs. 200 x_2

Total cost = 500 x_1 + 200 x_2 ,which cannot be more than 10000

$500\, x_1 + 200\, x_2 \leq 10000$

$5\, x_1 + 2x_2 \leq 100$

(iv) **Non-negative restrictions:** Since the number of tables and chairs cannot be negative, we have $x_1 \geq 0$, $x_2 \geq 0$

Thus, the mathematical formulation of the LPP is

Maximize $Z=50\, x_1+15x_2$

Subject to the constrains

$$x_1+x_2 \leq 60$$

$$5x_1+2x_2 \leq 100$$

$$x_1,\, x_2 \geq 0$$

Example 10.2

A company is producing three products P_1, P_2 and P_3, with profit contribution of Rs.20,Rs.25 and Rs.15 per unit respectively. The resource requirements per unit of each of the products and total availability are given below.

Product	P_1	P_2	P_3	Total availability
Man hours/unit	6	3	12	200
Machine hours/unit	2	5	4	350
Material/unit	1kg	2kg	1kg	100kg

Formulate the above as a linear programming model.

Solution:

(i) **Variables:** Let x_1 , x_2 and x_3 be the number of units of products P_1, P_2 and P_3 to be produced.

(ii) **Objective function:** Profit on x_1 units of the product $P_1 = 20\, x_1$

Profit on x_2 units of the product $P_2 = 25\, x_2$

Profit on x_3 units of the product $P_3 = 15\, x_3$

Total profit $= 20\, x_1 +25\, x_2 +15\, x_3$

Since the total profit is to be maximized, we have to maximize $Z= 20\, x_1+25\, x_2 +15\, x_3$

Constraints:

$$6x_1 +3\, x_2 +12\, x_3 \leq 200$$

$$2\, x_1 +5\, x_2 +4\, x_3 \leq 350$$

$$x_1 +2\, x_2 + x_3 \leq 100$$

Non-negative restrictions: Since the number of units of the products A,B and C cannot be negative, we have $x_1, x_2, x_3 \geq 0$

Thus, we have the following linear programming model.

Maximize $Z = 20\, x_1 + 25\, x_2 + 15\, x_3$

Subject to $6\, x_1 + 3\, x_2 + 12\, x_3 \leq 200$

$2x_1 + 5x_2 + 4\, x_3 \leq 350$

$x_1 + 2\, x_2 + x_3 \leq 100$

$x_1, x_2, x_3 \geq 0$

Example 10.3

A dietician wishes to mix two types of food F_1 and F_2 in such a way that the vitamin contents of the mixture contains atleast 6units of vitamin A and 9 units of vitamin B. Food F_1 costs Rs.50 per kg and F_2 costs Rs 70 per kg. Food F_1 contains 4 units per kg of vitamin A and 6 units per kg of vitamin B while food F_2 contains 5 units per kg of vitamin A and 3 units per kg of vitamin B. Formulate the above problem as a linear programming problem to minimize the cost of mixture.

Solution:

(i) **Variables:** Let the mixture contains x_1 kg of food F_1 and x_2 kg of food F_2

(ii) **Objective function:** cost of x_1 kg of food $F_1 = 50\, x_1$

cost of x_2 kg of food $F_2 = 70x_2$

The cost is to be minimized

Therefore minimize $Z = 50\, x_1 + 70x_2$

(iii) **Constraints:**

We make the following table from the given data

Resources	Food (in kg)		Requirement
	$F_1\ (x_1)$	$F_2\ (x_2)$	
Vitamin A (units/kg)	4	5	6
Vitamin B (units/kg)	6	3	9
Cost (Rs/kg)	50	70	

Table 10.1

$4x_1+5x_2 \geq 6$ (since the mixture contains 'atleast 6' units of vitamin A ,we have the inequality of the type $\geq$)

$6x_1+3x_2 \geq 9$(since the mixture contains 'atleast 9' units of vitamin B ,we have the inequality of the type $\geq$)

(iv) Non-negative restrictions:

Since the number of kgs of vitamin A and vitamin B are non-negative , we have $x_1, x_2 \geq 0$

Thus, we have the following linear programming model

$$\text{Minimize } Z = 50x_1 + 70x_2$$

$$\text{subject to} \quad 4x_1+5x_2 \geq 6$$

$$6x_1+3x_2 \geq 9$$

$$\text{and } x_1, x_2 \geq 0$$

Example 10.4

A soft drink company has two bottling plants C_1 and C_2. Each plant produces three different soft drinks S_1,S_2 and S_3. The production of the two plants in number of bottles per day are:

Product	Plant	
	C_1	C_2
S_1	3000	1000
S_2	1000	1000
S_3	2000	6000

A market survey indicates that during the month of April there will be a demand for 24000 bottles of S_1, 16000 bottles of S_2 and 48000 bottles of S_3. The operating costs, per day, of running plants C_1 and C_2 are respectively Rs.600 and Rs.400. How many days should the firm run each plant in April so that the production cost is minimized while still meeting the market demand? Formulate the above as a linear programming model.

Solution:

(i) **Variables:** Let x_1 be the number of days required to run plant C_1

and x_2 be the number of days required to run plant C_2

Objective function: Minimize $Z = 600x_1 + 400x_2$

(ii) **Constraints:** $3000\ x_1 + 1000\ x_2 \geq 24000$(since there is a demand of 24000 bottles of drink A, production should not be less than 24000)

$$1000\ x_1 + 1000\ x_2 \geq 16000$$

$$2000\ x_1 + 6000\ x_2 \geq 48000$$

(iii) Non-negative restrictions: Since be the number of days required of a firm are non-negative, we have $x_1, x_2 \geq 0$

Thus we have the following LP model.

Minimize $\quad Z = 600\ x_1 + 400\ x_2$

subject to $\quad 3000\ x_1 + 1000\ x_2 \geq 24000$

$1000\ x_1 + 1000\ x_2 \geq 16000$

$2000\ x_1 + 6000\ x_2 \geq 48000$

and $x_1, x_2 \geq 0$

10.1.2 Solution of LPP by graphical method

After formulating the linear programming problem, our aim is to determine the values of decision variables to find the optimum (maximum or minimum) value of the objective function. Linear programming problems which involve only two variables can be solved by graphical method. If the problem has three or more variables, the graphical method is impractical.

The major steps involved in this method are as follows

(i) State the problem mathematically

(ii) Write all the constraints in the form of equations and draw the graph

(iii) Find the feasible region

(iv) Find the coordinates of each vertex (corner points) of the feasible region. The coordinates of the vertex can be obtained either by inspection or by solving the two equations of the lines intersecting at the point

(v) By substituting these corner points in the objective function we can get the values of the objective function

(vi) If the problem is maximization then the maximum of the above values is the optimum value. If the problem is minimization then the minimum of the above values is the optimum value

Example 10.5

Solve the following LPP

Maximize $Z = 2\,x_1 + 5x_2$

subject to the conditions $x_1 + 4x_2 \leq 24$

$3x_1 + x_2 \leq 21$

$x_1 + x_2 \leq 9$ and $x_1,\ x_2 \geq 0$

Solution:

First we have to find the feasible region using the given conditions.

Since both the decision variables x_1 and x_2 are non-negative ,the solution lies in the first quadrant.

Write all the inequalities of the constraints in the form of equations.

Therefore we have the lines $x_1 + 4x_2 = 24$; $3x_1 + x_2 = 21$; $x_1 + x_2 = 9$

$x_1 + 4x_2 = 24$ is a line passing through the points (0 , 6) and (24 , 0).

[(0,6) is obtained by taking $x_1 = 0$ in $x_1 + 4x_2 = 24$, (24 , 0) is obtained by taking $x_2 = 0$ in $x_1 + 4x_2 = 24$].

Any point lying on or below the line $x_1 + 4x_2 = 24$ satisfies the constraint $x_1 + 4x_2 \leq 24$.

$3\,x_1 + x_2 = 21$ is a line passing through the points (0, 21) and (7, 0). Any point lying on or below the line $3\,x_1 + x_2 = 21$ satisfies the constraint $3\,x_1 + x_2 \leq 21$.

$x_1 + x_2 = 9$ is a line passing through the points (0 , 9) and (9 , 0) .Any point lying on or below the line $x_1 + x_2 = 9$ satisfies the constraint $x_1 + x_2 \leq 9$.

Now we draw the graph.

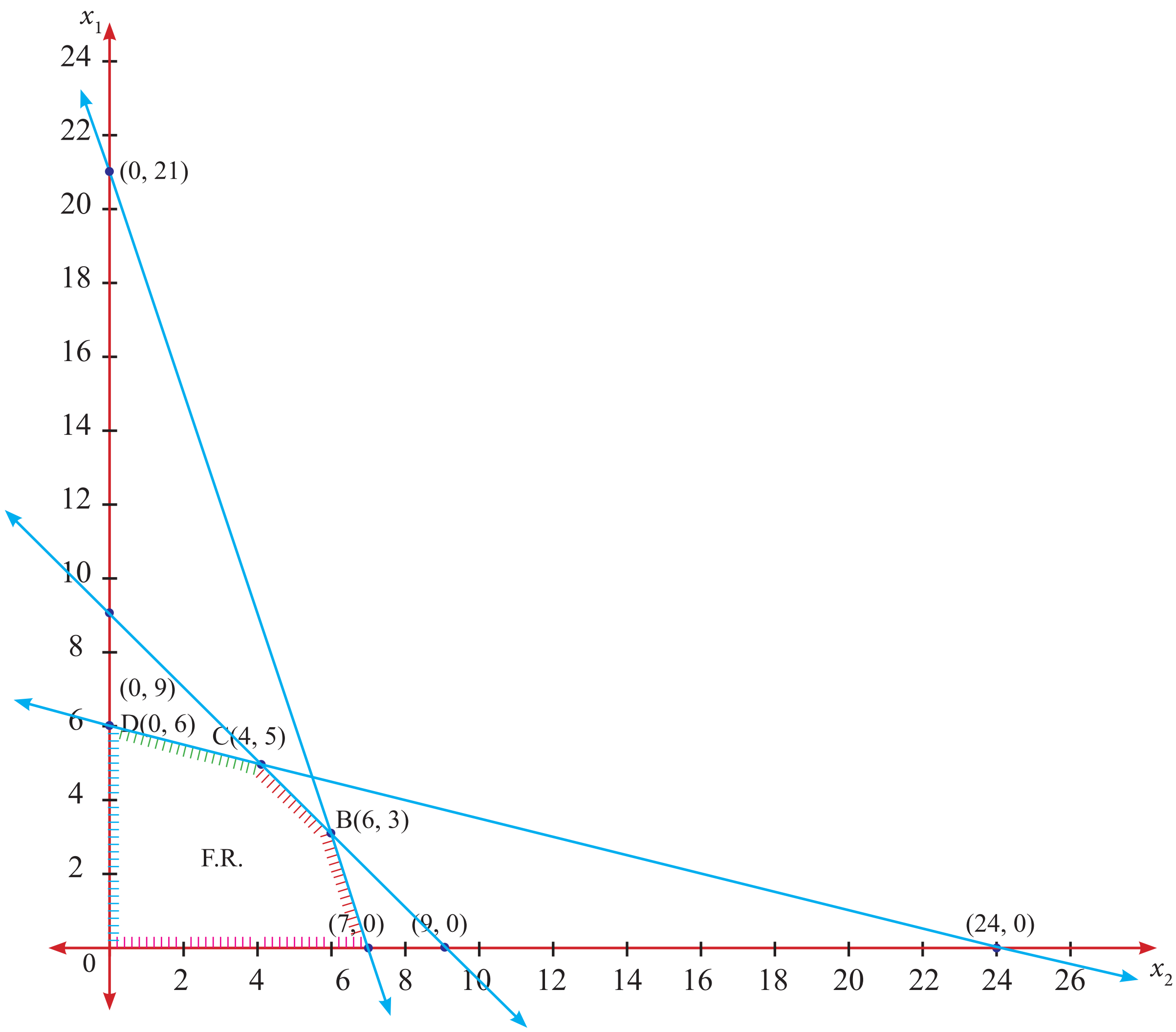


Fig 10.1

The feasible region satisfying all the conditions is OABCD.The co-ordinates of the points are O(0,0) A(7,0);B(6,3) [the point B is the intersection of two lines $x_1 + x_2 = 9$ and $3x_1 + x_2 = 21$];C(4,5) [the point C is the intersection of two lines

$x_1 + x_2 = 9$ and $x_1 + 4x_2 = 24$] and D(0,6).

Corner points	$Z = 2x_1 + 5x_2$
O(0,0)	0
A(7,0)	14
B(6,3)	27
C(4,5)	33
D(0,6)	30

Table 10.2

Maximum value of Z occurs at C. Therefore the solution is $x_1 = 4$, $x_2 = 5$, $Z_{max} = 33$

Example 10.6

Solve the following LPP by graphical method Minimize $z = \mathbf{5}x_{\mathbf{1}}+\mathbf{4}x_2$ Subject to constraints $4x_1+ x_2 \geq 40$; $2x_1+3x_2 \geq 90$ and $x_1, x_2 \geq 0$

Solution:

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations $4x_1+x_2 = 40$ and $2x_1+3x_2 = 90$

$4x_1+x_2 = 40$ is a line passing through the points (0,40) and (10,0).Any point lying on or above the line $4x_1+x_2= 40$ satisfies the constraint $4x_1+ x_2 \geq 40$.

$2x_1+3x_2 = 90$ is a line passing through the points (0,30) and (45,0). Any point lying on or above the line $2x_1+3x_2= 90$ satisfies the constraint $2x_1+3x_2 \geq 90$.

Draw the graph using the given constraints.

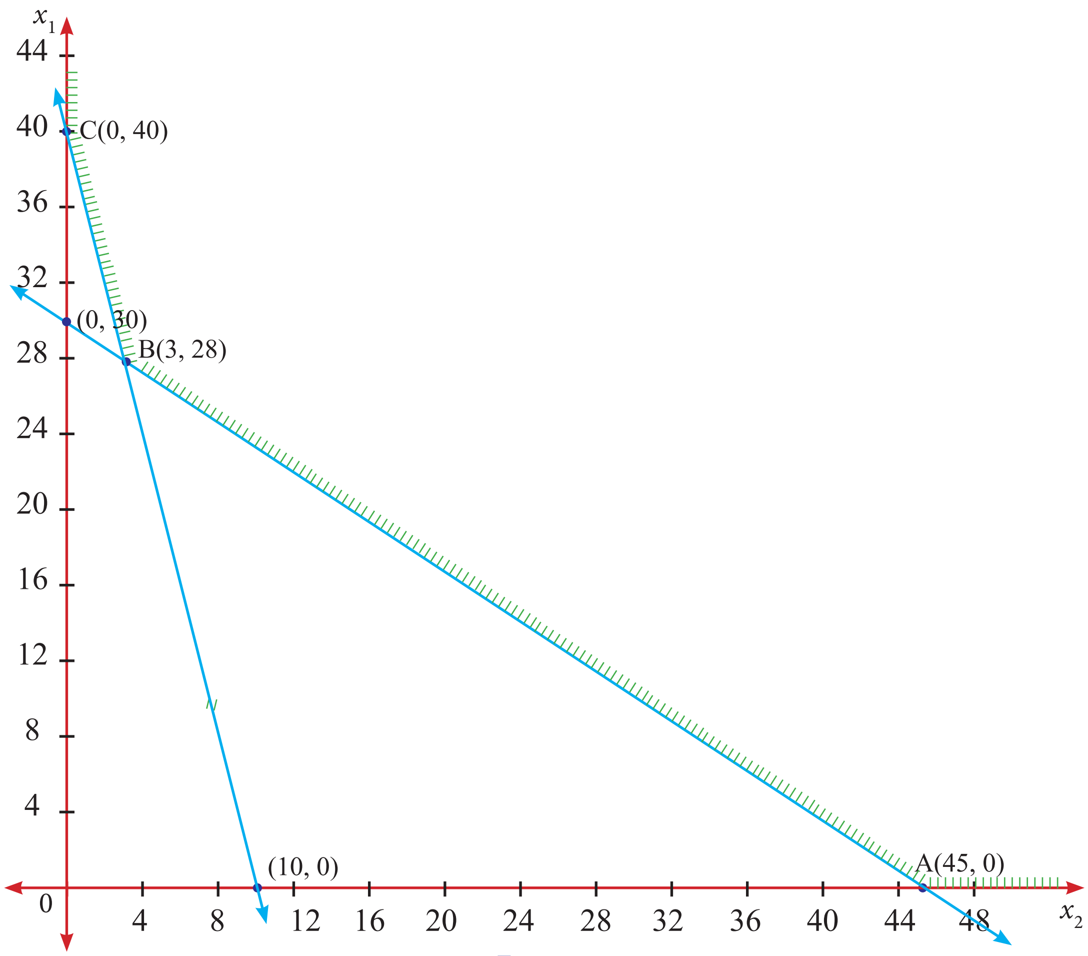


Fig 10.2

The feasible region is ABC (since the problem is of minimization type we are moving towards the origin.

Corner points	$z = 5x_1+4x_2$
A(45,0)	225
B(3,28)	127
C(0,40)	160

Table 10.4

The minimum value of Z occurs at B(3,28).

Hence the optimal solution is $x_1 = 3$, $x_2 = 28$ and $Z_{min}=127$

Example 10.7

Solve the following LPP.

Maximize $Z= 2\ x_1 +3x_2$ subject to constraints $x_1 + x_2 \leq 30$; $x_2 \leq 12$; $x_1 \leq 20$ and $x_1, x_2 \geq 0$

Solution:

We find the feasible region using the given conditions.

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Write all the inequalities of the constraints in the form of equations.

Therefore we have the lines

$$x_1+x_2=30;\ x_2 =12;\ x_1= 20$$

x_1+x_2 =30 is a line passing through the points (0,30) and (30,0)

x_2 = 12 is a line parallel to x_1–axis

x_1 = 20 is a line parallel to x_2–axis.

The feasible region satisfying all the conditions $x_1+ x_2\leq 30$; $x_2\leq 12$; $x_1\leq 20$ and $x_1, x_2 \geq 0$ is shown in the following graph.

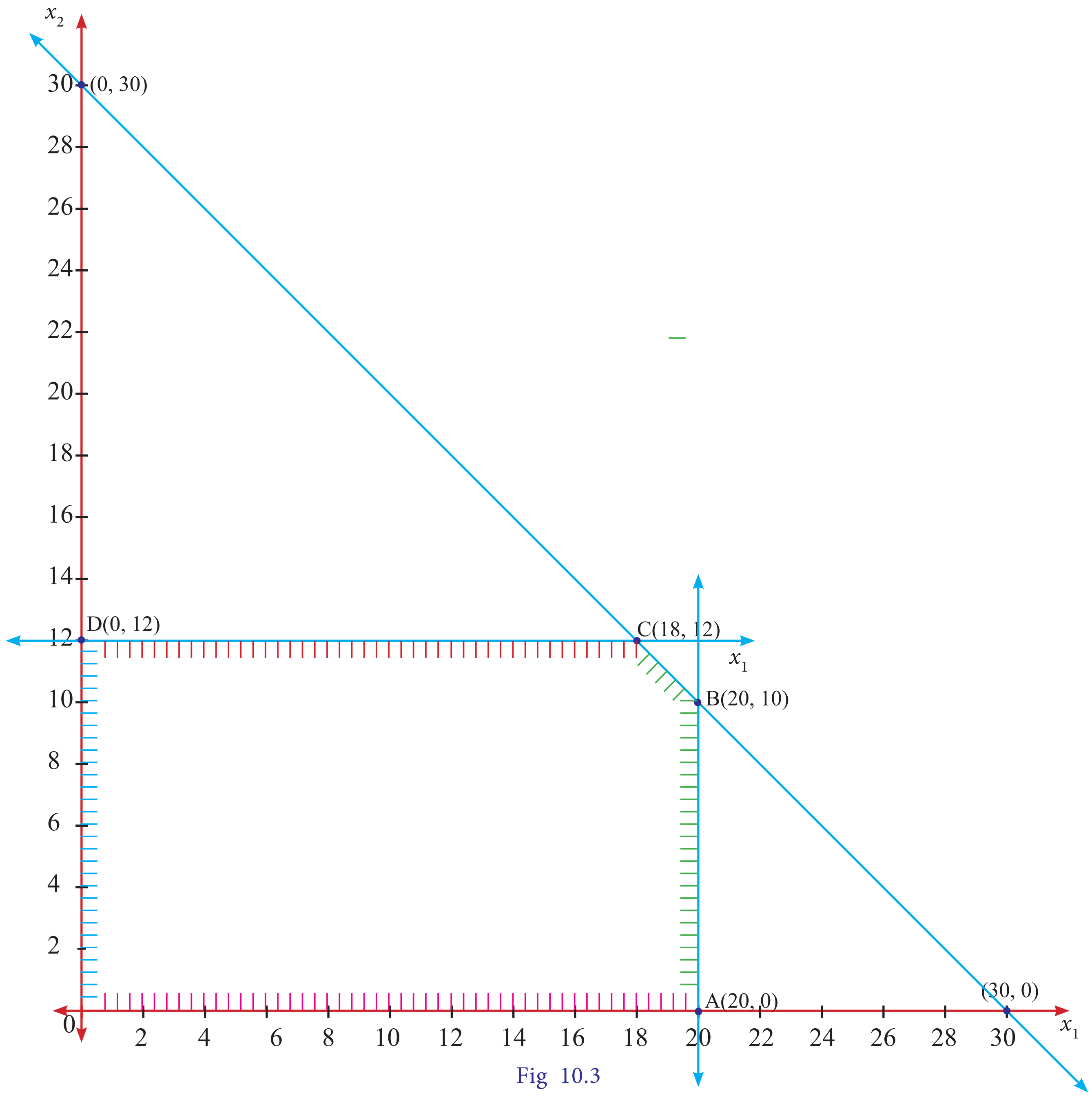


Fig 10.3

The feasible region satisfying all the conditions is OABCD.

The co-ordinates of the points are O(0,0) ; A(20,0); B(20,10) ; C(18,12) and D(0,12).

Corner points	$Z = 2x_1 + 3x_2$
O(0,0)	0
A(20,0)	40
B(20,10)	70
C(18,12)	72
D(0,12)	36

Table 10.3

Maximum value of Z occurs at C. Therefore the solution is $x_1 = 18$, $x_2 = 12$, $Z_{max} = 72$

Example 10.8

Maximize $Z = 3x_1 + 4x_2$ subject to $x_1 - x_2 \leq -1$; $-x_1+x_2 \leq 0$ and $x_1, x_2 \geq 0$

Solution:

Since both the decision variables x_1, x_2 are non-negative ,the solution lies in the first quadrant of the plane.

Consider the equations $x_1 - x_2 = -1$ and $-x_1 + x_2 = 0$

$x_1 - x_2 = -1$ is a line passing through the points (0,1) and (–1,0)

$-x_1 + x_2 = 0$ is a line passing through the point (0,0)

Now we draw the graph satisfying the conditions $x_1 - x_2 \leq -1$; $-x_1+x_2 \leq 0$ and $x_1, x_2 \geq 0$

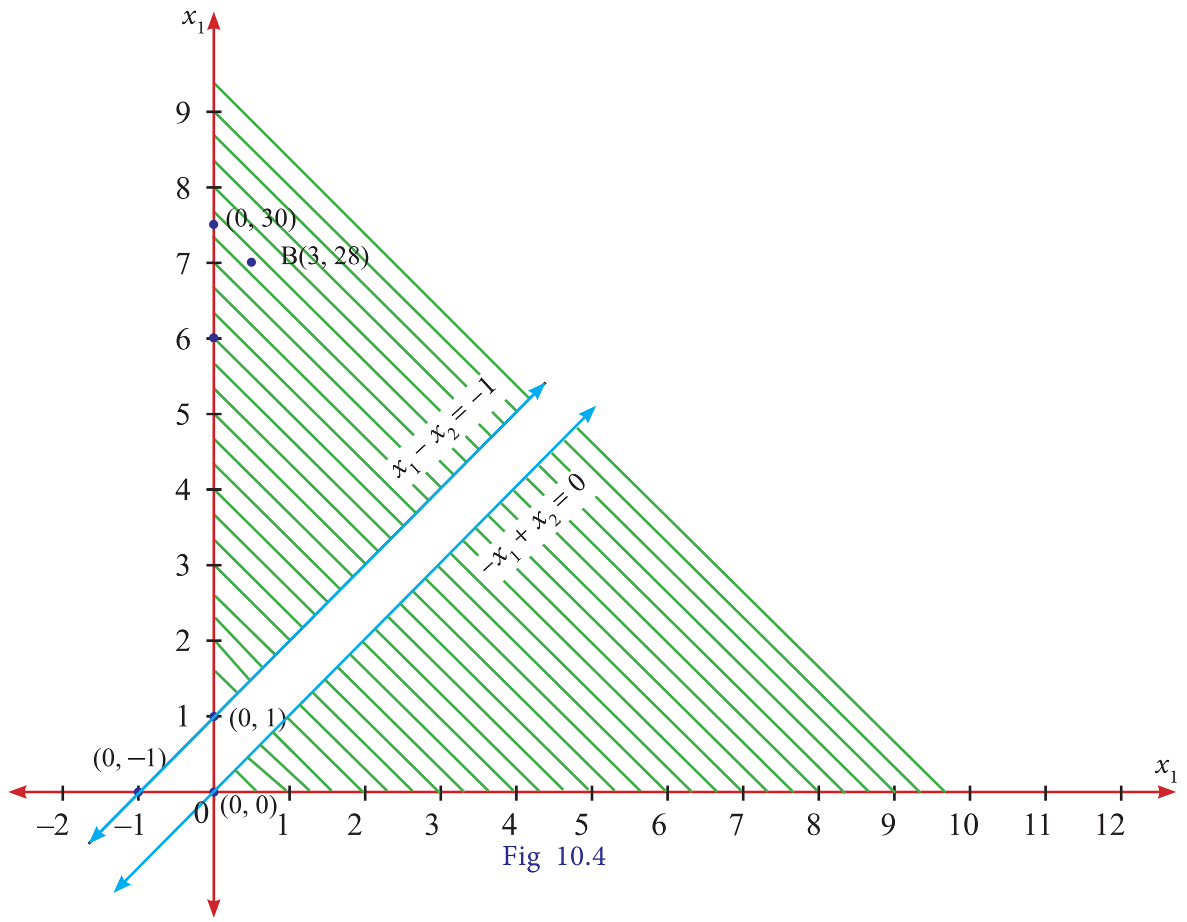


Fig 10.4

There is no common region(feasible region) satisfying all the given conditions. Hence the given LPP has no solution.

Exercise 10.1

1. A company produces two types of pens A and B. Pen A is of superior quality and pen B is of lower quality . Profits on pens A and B are Rs 5 and Rs 3 per pen respectively. Raw materials required for each pen A is twice as that of pen B. The supply of raw material is sufficient only for 1000 pens per day . Pen A requires a special clip and only 400 such clips are available per day. For pen B, only 700 clips are available per day . Formulate this problem as a linear programming problem.

2. A company produces two types of products say type A and B. Profits on the two types of product are Rs.30/- and Rs.40/- per kg respectively. The data on resources required and availability of resources are given below.

	Requirements		Capacity available per month
	Product A	Product B	
Raw material (kgs)	60	120	12000
Machining hours / piece	8	5	600
Assembling (man hours)	3	4	500

Formulate this problem as a linear programming problem to maximize the profit.

3. A company manufactures two models of voltage stabilizers viz., ordinary and auto-cut. All components of the stabilizers are purchased from outside sources , assembly and testing is carried out at company's own works. The assembly and testing time required for the two models are 0.8 hour each for ordinary and 1.20 hours each for auto-cut. Manufacturing capacity 720 hours at present is available per week. The market for the two models has been surveyed which suggests maximum weekly sale of 600 units of ordinary and 400 units of auto-cut . Profit per unit for ordinary and auto-cut models has been estimated at Rs 100 and Rs 150 respectively. Formulate the linear programming problem.

4. Solve the following linear programming problems by graphical method.

(i) Maximize $Z = 6x_1 + 8x_2$ subject to constraints $30x_1+20x_2 \leq 300; 5x_1+10x_2 \leq 110$; and $x_1, x_2 \geq 0$.

(ii) Maximize $Z = 22x_1 + 18x_2$ subject to constraints $960x_1 + 640x_2 \leq 15360$; $x_1 + x_2 \leq 20$ and $x_1, x_2 \geq 0$.

(iii) Minimize $Z = 3x_1 + 2x_2$ subject to the constraints $5x_1 + x_2 \geq 10$; $x_1 + x_2 \geq 6$; $x_1 + 4x_2 \geq 12$ and $x_1, x_2 \geq 0$.

(iv) Maximize $Z = 40x_1 + 50x_2$ subject to constraints $30x_1 + x_2 \leq 9$; $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$

(v) Maximize $Z = 20x_1 + 30x_2$ subject to constraints $3x_1 + 3x_2 \leq 36$; $5x_1 + 2x_2 \leq 50$; $2x_1 + 6x_2 \leq 60$ and $x_1, x_2 \geq 0$

(vi) Minimize $Z = 20x_1 + 40x_2$ subject to the constraints $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$ and $x_1, x_2 \geq 0$

10.2 Network Analysis

Networks are diagrams easily visualized in transportation system like roads, railway lines, pipelines, blood vessels, etc.

A project will consist of a number of jobs and particular jobs can be started only after finishing some other jobs. There may be jobs which may not depend on some other jobs. Network scheduling is a technique which helps to determine the various sequences of jobs concerning a project and the project completion time. There are two basic planning and control techniques that use a network to complete a pre-determined schedule. They are Program Evaluation and Review Technique (PERT) and Critical Path Method (CPM). The critical path method (CPM) was developed in 1957 by JE Kelly of Ramington R and M.R. Walker of Dupon to help schedule maintenance of chemical plants. CPM technique is generally applied to well known projects where the time schedule to perform the activities can exactly be determined.

Some important definitions in network

Activity:

An activity is a task or item of work to be done, that consumes time, effort, money or other resources. It lies between two events, called the starting event and ending event. An activity is represented by an arrow indicating the direction in which the events are to occur.

Event:

The beginning and end points of an activity are called events (or nodes). Event is a point in time and does not consume any resources. The beginning and completion of an activity are known as tail event and head event respectively. Event is generally represented by a numbered circle .The head event, called the j^{th} event, has always a number higher than the tail event, called the i^{th} event, ie., $j > i$.

Predecessor Activity:

Activities which must be completed before a particular activity starts are called predecessor activities. If an activity A is predecessor of an activity B, it is denoted by $A<B$. (i.e.,) activity B can start only if activity A is completed.

Successor Activity:

An activity that cannot be started until one or more of the other activities are completed, but immediately succeed them is referred to as successor activity.

Network:

Network is a diagrammatic representation of various activities concerning a project arranged in a logical manner.

Path:

A path is defined as a set of nodes connected by lines which begin at the initial node and end at the terminal node of the network.

10.2.1 Construction of network:

Rules for constructing network

For the construction of a network, generally, the following rules are followed:

(i) Each activity is represented by one and only one arrow.(i.e) only one activity can connect any two nodes.

(ii) No two activities can be identified by the same head and tail events.

(iii) Nodes are numbered to identify an activity uniquely. Tail node (starting point) should be lower than the head node (end point) of an activity.

(iv) Arrows should not cross each other.

(v) Arrows should be kept straight and not curved or bent.

(vi) Every node must have atleast one activity preceding it and atleast one activity following it except for the node at the beginning and at the end of the network.

Numbering the Events

After the network is drawn in a logical sequence, every event is assigned a number. The number sequence must be such as to reflect the flow of the network. In event numbering, the following rules should be observed:

(i) Event numbers should be unique.

(ii) Event numbering should be carried out on a sequential basis from left to right.

(iii) The initial event is numbered 0 or 1.

(iv) The head of an arrow should always bear a number higher than the one assigned at the tail of the arrow.

(v) Gap should be left in the sequence of event numbering to accommodate subsequent inclusion of activities, if necessary.

Remark: The above procedure of assigning numbers to various events of a network is known as Fulkerson's Rule.

Example 10.9

Draw the logic network for the following:

Activities C and D both follow A , activity E follows C , activity F follows D , activity E and F precedes B.

Solution:

The required network for the above information.

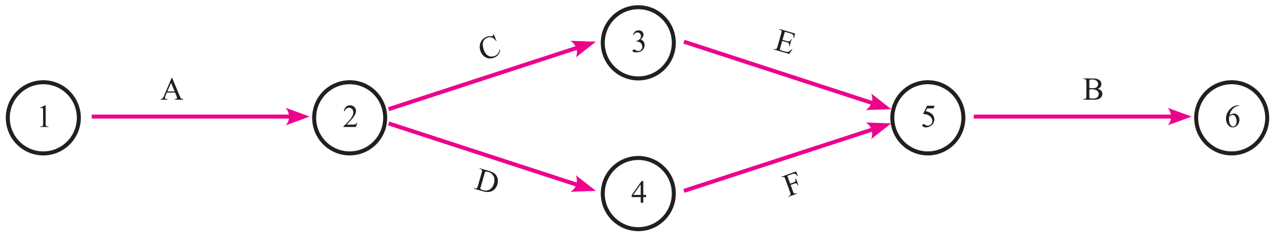


Fig 10.5

Example 10.10

Develop a network based on the following information:

Activity :	A	B	C	D	E	F	G	H
Immediate predecessor:	-	-	A	B	C,D	C,D	E	F

Solution:

Using the immediate precedence relationships and following the rules of network construction, the required network is shown in following figure

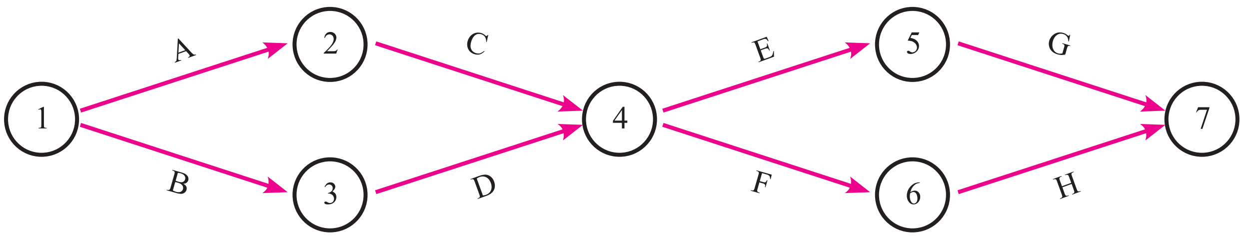


Fig 10.6

Dummy activity:

An activity which does not consume any resource or time, but merely depict the technological dependence is called a dummy activity. It is represented by dotted lines.

Example 10.11

Draw a network diagram for the project whose activities and their predecessor relationships are given below:

Activity :	A	B	C	D	E	F	G	H	I	J	K
Predecessor activity:	-	-	-	A	B	B	C	D	F	H,I	F,G

Solution:

Using the precedence relationships and following the rules of network construction, the required network diagram is shown in following figure

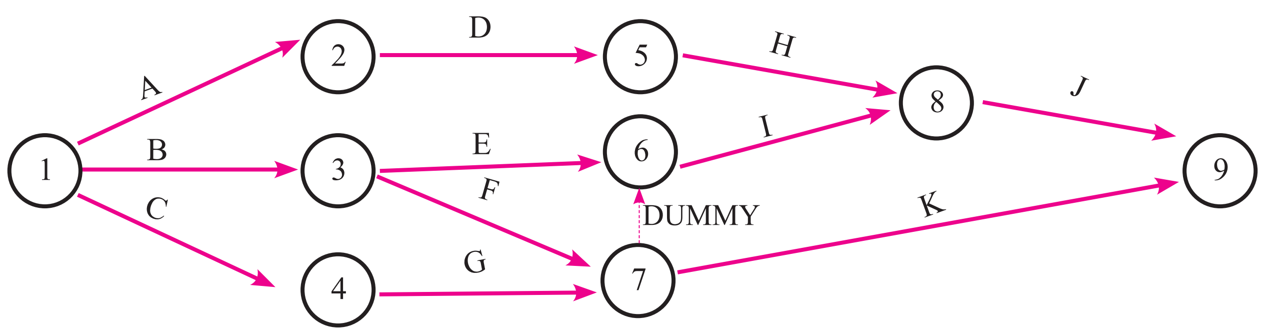


Fig 10.7

Example 10.12

Construct a network diagram for the following situation:

$A<D,E;\quad B, D<F;\quad C<G$ and $B<H.$

Solution:

Using the precedence relationships and following the rules of network construction, the required network is shown in following figure

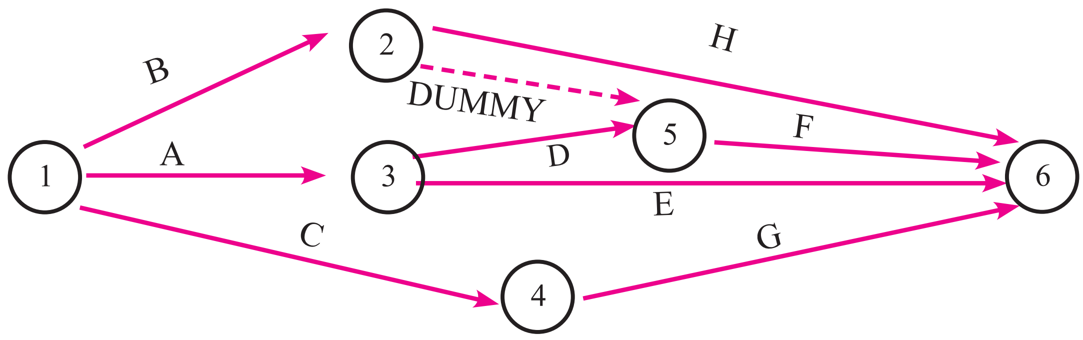


Fig 10.8

10.2.2 Critical path analysis

For each activity an estimate must be made of time that will be spent in the actual accomplishment of that activity. Estimates may be expressed in hours, days, weeks or any other convenient unit of time. The time estimate is usually written in the network immediately above the arrow. For the purpose of calculating various times of events and activities, the following terms shall be used in critical path calculations:

E_i = Earliest start time of event i

L_j = Latest start time of event j

t_{ij} = Duration of activity (i,j)

The next step after making the time estimates is the calculations which are done in the following ways:

(i) Forward Pass Calculations

(ii) Backward Pass Calculations.

Forward pass calculations:

We start from the initial node 1 with starting time of the project as zero. Proceed through the network visiting nodes in an increasing order of node number and end at the final node of the network. At each node, we calculate earliest start times for each activity by considering E_i as the earliest occurrence of node i.

The method may be summarized as below:

Step 1. Set $E_1 = 0$; $i = 1$ (initial node)

Step 2. Set the earliest start time(EST) for each activity that begins at node i as $ES_{ij} = E_i$; for all activities (i, j) that start at node i.

Step 3. Compute the earliest finish time(EFT) of each activity that begins at node i by adding the earliest start time of the activity to the duration of the activity. Thus $EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$

Step 4. Move on to next node, say node $j(j > i)$ and compute the earliest start time at node j, using $E_j = \max i\{EF_{ij}\} = \max_i\{E_i + t_{ij}\}$ for all immediate predecessor activities.

Step 5. If $j = n$ (final node), then the earliest finish time for the project is given by $E_n = \max\{EF_{ij}\} = \max\{E_{n-1} + t_{ij}\}$

Backward pass calculations:

We start from the final (last) node n of the network, proceed through the network visiting nodes in the decreasing order of node numbers and end at the initial node 1. At each node, we calculate the latest finish time and latest start time for each activity by considering L_j as the latest occurrence of node j. The method may be summarized below:

Step 1. $L_n = E_n$; for $j = n$

Step 2. Set the latest finish time (LFT)of each activity that ends at node j as $LF_{ij} = L_j$

Step 3. Compute the latest start time(LST) of all activities ending at node j, subtracting the duration of each activity from the latest finish time of the activity. Thus, $LS_{ij} = LF_{ij} - t_{ij} = L_j - t_{ij}$

Step 4. Proceed backward to the next node i($i<j$) in the sequence and compute the latest occurrence time at node i using

$$L_i = \min_j\{LS_{ij}\} = \min_j\{L_j - t_{ij}\}$$

Step 5. If j = 1 (initial node), then $L_1 = \min\{LS_{ij}\} = \min\{L_2 - t_{ij}\}$

Critical path:

The longest path connected by the activities in the network is called the critical path. A path along which it takes the longest duration.

For the activity (i,j), to lie on the critical path, following conditions must be satisfied:

(i) $E_i = L_i$ and $E_j = L_j$ (ii) $E_j - E_i = L_j - L_i = t_{ij}$

Example 10.13

Compute the earliest start time, earliest finish time ,latest start time and latest finish time of each activity of the project given below:

Activity	1-2	1-3	2-4	2-5	3-4	4-5
Duration(in days)	8	4	10	2	5	3

Solution:

Earliest start time (EST)and latest finish time(LFT) of each activity are given ithe following network.

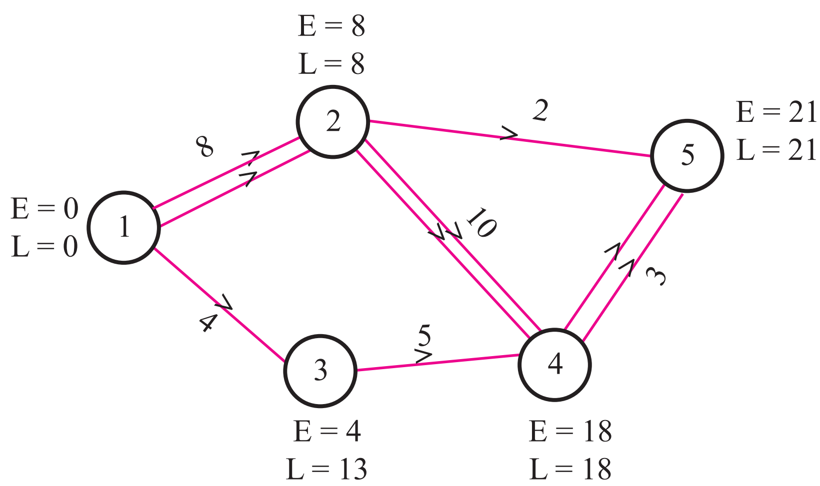


Fig 10.9

$E_1 = 0$

$E_2 = E_1 + t_{12} = 0 + 8 = 8$

$E_3 = E_1 + t_{13} = 0 + 4 = 4$

$E_4 = E_2 + t_{24}$ or $E_3 + t_{34} = 8 + 10 = 18$

(take $E_2 + t_{24}$ or $E_3 + t_{34}$ whichever is maximum)

$E_5 = (E_2 + t_{25}$ or $E_4 + t_{45}) = 18 + 3 = 21$

(take $E_2 + t_{25}$ or $E_4 + t_{45}$ whichever is maximum)

$L_5 = 21$

$L_4 = L_5 - t_{45} = 21 - 3 = 18$

$L_3 = L_4 - t_{34} = 18 - 5 = 13$

$L_2 = L_5 - t_{25}$ or $L_4 - t_{24} = 18 - 10 = 8$

(take $L_5 - t_{25}$ or $L_4 - t_{24}$ whichever is minimum)

$L_1 = L_2 - t_{12}$ or $L_3 - t_{13} = 8 - 8 = 0$

(take $L_2 - t_{12}$ or $L_3 - t_{13}$ whichever is minimum)

Here the critical path is 1-2-4-5, which is denoted by double lines.

Activity	Duration (t_{ij})	EST	EFT=EST+t_{ij}	LST=LFT−t_{ij}	LFT
1–2	8	0	8	0	8
1–3	4	0	4	9	13
2–4	10	8	18	8	18
2–5	2	8	10	19	21
3–4	5	4	9	13	18
4–5	3	18	21	18	21

Table 10.4

The longest duration to complete this project is 21 days.

The path connected by the critical activities is the critical path(the longest path).

Critical path is 1-2-4-5 and project completion time is 21 days.

Example 10.14

Calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity of the project given below and determine the Critical path of the project and duration to complete the project.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration (in week)	8	7	12	4	10	3	5	10	7	4

Solution:

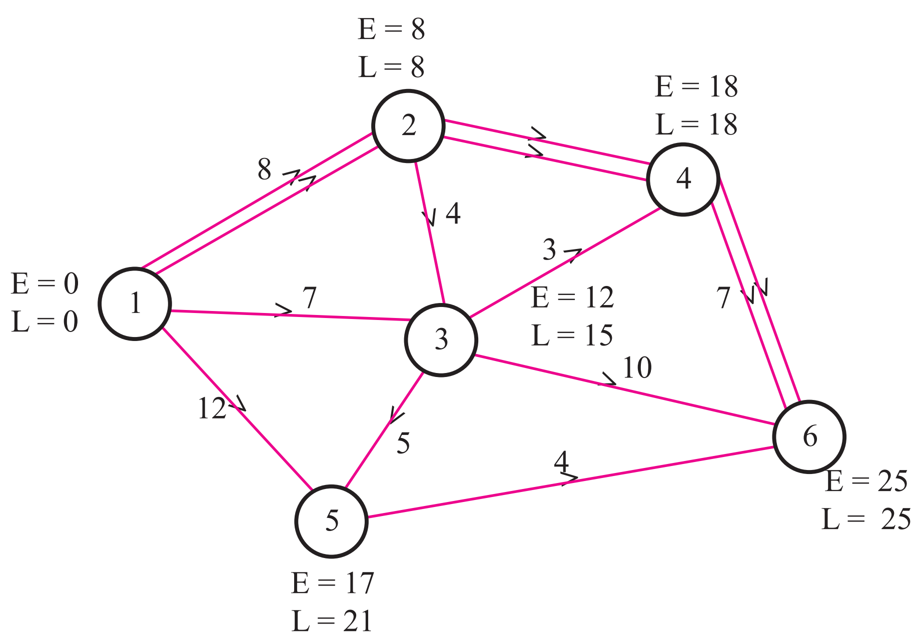


Fig 10.10

Activity	Duration (in week)	EST	EFT	LST	LFT
1-2	8	0	8	0	8
1-3	7	0	7	8	15
1-5	12	0	12	9	21
2-3	4	8	12	11	15
2-4	10	8	18	8	18
3-4	3	12	15	15	18
3-5	5	12	17	16	21
3-6	10	12	22	15	25
4-6	7	18	25	18	25
5-6	4	17	21	21	25

Table 10.5

Here the critical path is 1–2–4–6

The project completion time is 25 weeks

Exercise 10.2

1. Draw the network for the project whose activities with their relationships are given below:

   Activities A,D,E can start simultaneously; B,C>A; G,F>D,C; H>E,F.

2. Draw the event oriented network for the following data:

Events	1	2	3	4	5	6	7
Immediate Predecessors	-	1	1	2,3	3	4,5	5,6

3. Construct the network for the projects consisting of various activities and their precedence relationships are as given below:

   A,B,C can start simultaneously A<F, E; B<D, C; E, D<G

4. Construct the network for each the projects consisting of various activities and their precedence relationships are as given below:

Activity	A	B	C	D	E	F	G	H	I	J	K
Immediate Predecessors	-	-	-	A	B	B	C	D	E	H,I	F,G

5. Construct the network for the project whose activities are given below.

Activity	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration(in week)	3	8	12	6	3	3	8	5	3	8

   Calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity. Determine the critical path and the project completion time.

6. A project schedule has the following characteristics

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8	8-10	9-10
Time	4	1	1	1	6	5	4	8	1	2	5	7

   Construct the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

7. Draw the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

Jobs	1-2	1-3	2-4	3-4	3-5	4-5	4-6	5-6
Duration	6	5	10	3	4	6	2	9

8. The following table gives the activities of a project and their duration in days

Activity	1-2	1-3	2-3	2-4	3-4	3-5	4-5
Duration	5	8	6	7	5	4	8

Construct the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

9. A Project has the following time schedule

Activity	1-2	1-6	2-3	2-4	3-5	4-5	6-7	5-8	7-8
Duration(in days)	7	6	14	5	11	7	11	4	18

Construct the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

10. The following table use the activities in a construction projects and relevant information

Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration (in days)	22	27	12	14	6	12

Draw the network for the project ,calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and find the critical path. Compute the project duration

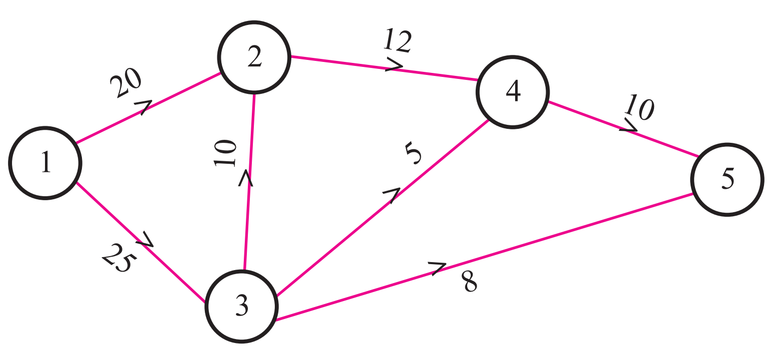
ILR3FE

Exercise 10.3

Objective type questions

Choose the correct answer

1. The critical path of the following network is



(a) 1 – 2 – 4 – 5 (b) 1– 3– 5 (c) 1 – 2 – 3 – 5 (d) 1 – 2 – 3 – 4 – 5

2. Maximize: $z = 3x_1 + 4x_2$ subject to $2x_1 + x_2 \leq 40$, $2x_1 + 5x_2 \leq 180$, $x_1, x_2 \geq 0$. In the LPP, which one of the following is feasible corner point ?

(a) $x_1 = 18, x_2 = 24$ (b) $x_1 = 15, x_2 = 30$

(c) $x_1 = 2 \cdot 5, x_2 = 35$ (d) $x_1 = 20 \cdot 5, x_2 = 19$

3. One of the conditions for the activity (i, j) to lie on the critical path is

(a) $E_j - E_i = L_j - L_i = t_{ij}$ (b) $E_i - E_j = L_j - L_i = t_{ij}$

(c) $E_j - E_i = L_i - L_j = t_{ij}$ (d) $E_j - E_i = L_j - L_i \neq t_{ij}$

4. In constructing the network which one of the following statement is false ?

(a) Each activity is represented by one and only one arrow. (i.e) only one activity can connect any two nodes.

(b) Two activities can be identified by the same head and tail events.

(c) Nodes are numbered to identify an activity uniquely. Tail node (starting point) should be lower than the head node (end point) of an activity.

(d) Arrows should not cross each other.

5. In a network while numbering the events which one of the following statement is false ?

(a) Event numbers should be unique.

(b) Event numbering should be carried out on a sequential basis from left to right.

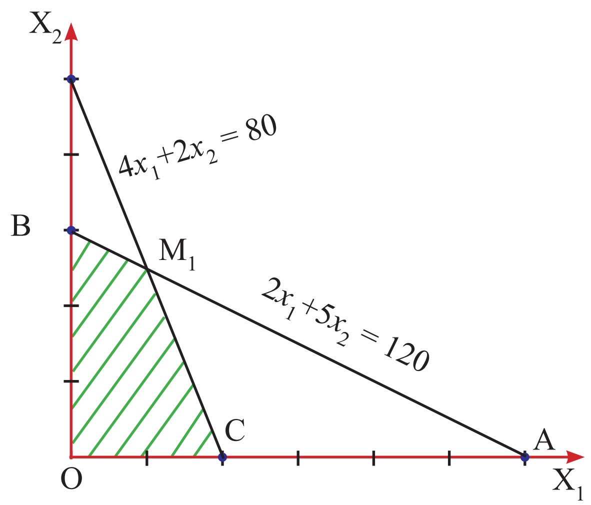
(c) The initial event is numbered 0 or 1.

(d) The head of an arrow should always bear a number lesser than the one assigned at the tail of the arrow.

6. A solution which maximizes or minimizes the given LPP is called

(a) a solution (b) a feasible solution (c) an optimal solution (d) none of these

7. In the given graph the coordinates of M_1 are



(a) $x_1 = 5, x_2 = 30$

(b) $x_1 = 20, x_2 = 16$

(c) $x_1 = 10, x_2 = 20$

(d) $x_1 = 20, x_2 = 30$

8. The maximum value of the objective function $Z = 3x + 5y$ subject to the constraints $x > 0, y > 0$ and $2x + 5y \leq 10$ is

(a) 6 (b) 15 (c) 25 (d) 31

9. The minimum value of the objective function $Z = x + 3y$ subject to the constraints $2x + y \leq 20, x + 2y \leq 20, x > 0$ and $y > 0$ is

(a) 10 (b) 20 (c) 0 (d) 5

10. Which of the following is not correct?

(a) Objective that we aim to maximize or minimize

(b) Constraints that we need to specify

(c) Decision variables that we need to determine

(d) Decision variables are to be unrestricted.

11. In the context of network, which of the following is not correct

(a) A network is a graphical representation .

(b) A project network cannot have multiple initial and final nodes

(c) An arrow diagram is essentially a closed network

(d) An arrow representing an activity may not have a length and shape

12. The objective of network analysis is to

(a) Minimize total project cost

(b) Minimize total project duration

(c) Minimize production delays, interruption and conflicts

(d) All the above

13. Network problems have advantage in terms of project

(a) Scheduling (b) Planning

(c) Controlling (d) All the above

14. In critical path analysis, the word CPM mean

(a) Critical path method (b) Crash project management

(c) Critical project management (d) Critical path management

15. Given an L.P.P maximize $Z = 2x_1 + 3x_2$ subject to the constrains $x_1 + x_2 \leq 1$, $5x_1 + 5x_2 \geq 0$ and $x_1 \geq 0$, $x_2 \geq 0$ using graphical method, we observe

(a) No feasible solution (b) unique optimum solution

(c) multiple optimum solution (d) none of these

Miscellaneous Problems

1. A firm manufactures two products A and B on which the profits earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, While B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hrs 30 minutes while M_2 is available for 10 hrs during any working day. Formulate this problem as a linear programming problem to maximize the profit.

2. A firm manufactures pills in two sizes A and B. Size A contains 2 mgs of aspirin, 5 mgs of bicarbonate and 1 mg of codeine. Size B contains 1 mg. of aspirin, 8 mgs. of bicarbonate and 6 mgs. of codeine. It is found by users that it requires atleast 12 mgs. of aspirin, 74 mgs.of bicarbonate and 24 mgs. of codeine for providing immediate relief. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LLP.

3. Solve the following linear programming problem graphically.

Maximise $Z = 4x_1 + x_2$ subject to the constraints $x_1 + x_2 \leq 50$; $3x_1 + x_2 \leq 90$ and $x_1 \geq 0$, $x_2 \geq 0$

4. Solve the following linear programming problem graphically.

Minimize $Z = 200x_1 + 500x_2$ subject to the constraints: $x_1 + 2x_2 \geq 10$;

$3x_1 + 4x_2 \leq 24$ and $x_1 \geq 0, x_2 \geq 0$

5. Solve the following linear programming problem graphically.

Maximize $Z = 3x_1 + 5x_2$ subject to the constraints: $x_1 + x_2 \leq 6$, $x_1 \leq 4$; $x_2 \leq 5$, and $x_1, x_2 \geq 0$

6. Solve the following linear programming problem graphically. Maximize $Z = 60x_1 + 15x_2$ subject to the constraints: $x_1 + x_2 \leq 50$; $3x_1 + x_2 \leq 90$ and $x_1, x_2 \geq 0$

7. Draw a network diagram for the following activities.

Activity code	A	B	C	D	E	F	G	H	I	J	K
Predecessor activity	–	A	A	A	B	C	C	C,D	E,F	G,H	I,J

8. Draw the network diagram for the following activities

Activity code	A	B	C	D	E	F	G
Predecessor activity	-	-	A	A	B	C	D,E

9. A Project has the following time schedule

Activity	1-2	2-3	2-4	3-5	4-6	5-6
Duration (in days)	6	8	4	9	2	7

Draw the network for the project ,calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and find the critical path. Compute the project duration.

10. The following table gives the characteristics of project

Activity	1-2	1-3	2-3	3-4	3-5	4-6	5-6	6-7
Duration (in days)	5	10	3	4	6	6	5	5

Draw the network for the project ,calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and find the critical path. Compute the project duration.

Summary

- Linear programming problem(LPP) is a mathematical modeling technique which is used to allocate limited available resources in order to optimize (maximize or minimize) the objective function.
- **Short form of LPP**

  Maximize or Minimize $Z = \sum_{j=1}^{n} c_j x_j$

  Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq (\text{or} = \text{or} \geq) b_i,\ i = 1, 2, 3, \ldots, m$

  and $x_j \geq 0$
- **Objective function:** A function $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ which is to be optimized (maximized or minimized) is called objective function.
- **Decision variable:** The decision variables are the variables we seek to determine to optimize the objective function. x_j, $j = 1,2,3,\ldots,n$, are the decision variables.
- **Solution:** A set of values of decision variables x_j, $j = 1,2,3,\ldots,n$ satisfying all the constraints of the problem is called a solution to that problem.
- **Feasible solution:** A set of values of the decision variables that satisfies all the constraints of the problem and non-negativity restrictions is called a feasible solution of the problem.
- **Optimal solution:** Any feasible solution which maximizes or minimizes the objective function is called an optimal solution.
- **Feasible region:** The common region determined by all the constraints including non-negative constraints $x_j \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.
- Linear programming problems which involve only two variables can be solved by graphical method.
- It should be noted that the optimal value of LPP occurs at the corner points of the feasible region
- Network is a diagrammatic representation of various activities concerning a project arranged in a logical manner.

- An activity is a task or item of work to be done, that consumes time, effort, money or other resources
- The beginning and end points of an activity are called events (or nodes).
- **Critical path:** The longest path connected by the activities in the network is called the critical path. A path along which it takes the longest duration.

GLOSSARY	
abstract	பண்பு தொகை
activity	செயல்பாடு
backward pass calculations	பின் நோக்கி செல்லும் கணக்கீடு
critical path analysis	தீர்வுக்கு உகந்த பகுப்பாய்வு
critical path method	தீர்வுக்கு உகந்த முறை
decision variables	தீர்மான மாறிகள்
dummy activities	ஒப்புக்கான செயல்
earliest start time	முன்கூட்டியே தொடங்கும் நேரம்
event	நிகழ்வு
feasible solution	ஏற்புடைய தீர்வு
forward pass calculations	முன் நோக்கி செல்லும் கணக்கீடு
head event	ஆரம்ப நிகழ்வு
latest start time	சமீபத்திய தொடங்கும் நேரம்
linear programming problem	நேரியல் திட்டமிடல் கணக்கு
logical sequence	தர்க்க தொடர் வரிசை
network analysis	வலையமைப்பு பகுப்பாய்வு
predecessor activity	முந்தைய செயல்
successor activity	பிந்தைய செயல்
tail event	இறுதி நிகழ்வு

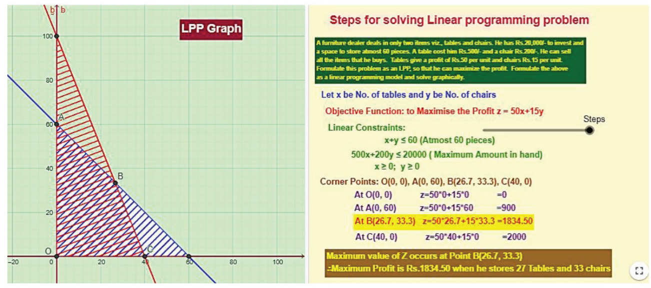
ICT Corner

Operational Research

Step – 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called “11th Business Maths Volume-2” will appear. In that there are several worksheets related to your Text Book.



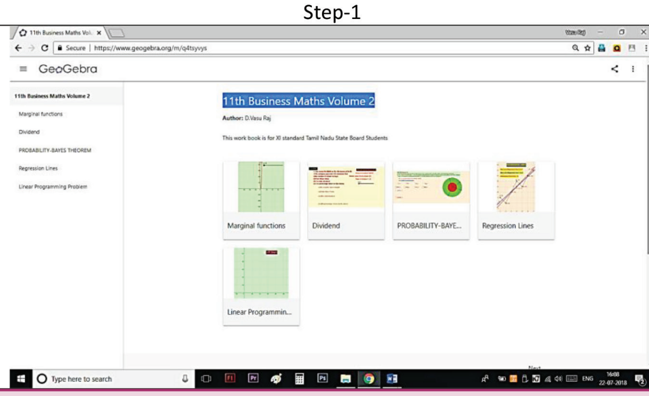
Expected Outcome ⇒

Step-2

Select the work sheet “Linear Programming Problem” Move the slider on Right side to see the steps for working Linear Programming Problem. Work out the problem as given. Graphical representation is given on left side. Also refer the worksheet “Inequality video” in the work book.

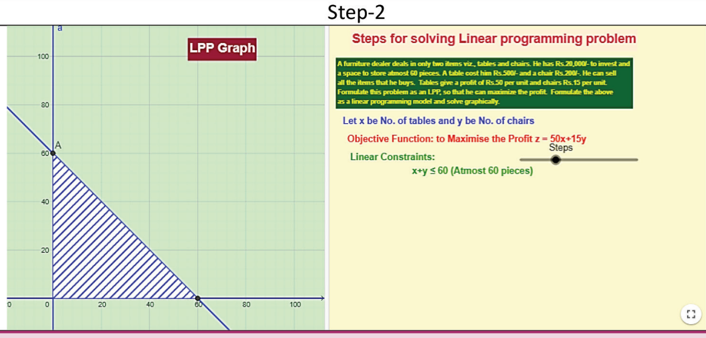
Step 1

Step-1



Step 2

Step-2



Browse in the link

11th Business Maths: https://ggbm.at/qKj9gSTG (or) scan the QR Code

B184_11_BM_EM

ANSWERS

6. Applications of differentiation

Exercise 6.1

1. $\quad , \; AVC = \frac{1}{10}x^2 - 4x - 20, \; AFC = \frac{7}{x}$

$MC = \frac{3}{10}x^2 - 8x - 20, \; MAC = \frac{2}{5}x - 4 - \frac{7}{x^2}$

2. $C = Rs.\frac{121}{16}, \; AC = Rs.\frac{29}{12}, \; MC = \frac{2}{3}$

3. $AC = x^2 - 2, \; MC = 3x^2 - 2, \; AR = 14 - x, \; MR = 14 - 2x$

4. $n_d = \frac{2}{x}$

5(i) $n_d = \frac{a - bx}{2bx}, \; x = \frac{a}{3b}$ (ii) $n_d = \frac{a - bx^2}{2bx^2}, x = \sqrt{\frac{a}{3b}}$

6. $\frac{4p^2}{2p^2+5}, \frac{36}{26}$

7. $MR = \frac{50-2x}{5}, \; 10, \; 0$

8. $\frac{p}{2\left(p-b\right)}, \; 1$

11. 4

12. $P = -\frac{x^2}{100} + 160x - 120, \; AP = \frac{-x}{100} + 160 - \frac{120}{x}, \; Rs\ 147.9$

$MP = \frac{-2x}{100} + 160, \; 159.8, \; MAP = -\frac{1}{100} + \frac{120}{x^2}, \; 1.19$

13. $x = -8, 2$

15. $n_d = \frac{p}{10-p}, \left|n_d\right| > 1 \Rightarrow$ elastic.

16. $P_E = 30, \; x_E = 40$

17. $x = 2100$ units , $p =$ Rs.130

18. $x = 6$ units

Exercise 6.2

1. AC is increasing when $x > 5$

3. P is maximum when $x = 46$, maximum profit Rs.1107.68

4. Revenue is maximum when $x = 220$

5. Minimum value is -71 , maximum value is 62

Exercise 6.3

1.

Items	EOQ in units	Total variable lost	EOQ in ₹	EOQ in year supply	Number if order per year
A	2000	₹4	40	2.5	0.4
B	200	₹20	200	0.5	2
C	2627	₹52.54	525.40	0.19	5.26

2 (i) 912 units per order (ii) ₹20, 065.80 per week.

Exercise 6.4

1. $\dfrac{\partial z}{\partial x} = a\left(cy + d\right), \dfrac{\partial z}{\partial y} = c\left(ax + b\right)$

Exercise 6.5

1. 23, 25 3. 2, 8 4. 0.8832 5. $-\dfrac{4}{3}, -8$ 6. $\dfrac{10}{79}, -\dfrac{3}{79}$

Exercise 6.6

1	2	3	4	5	6	7	8	9	10
(d)	(a)	(b)	(a)	(b)	(a)	(b)	(c)	(b)	(a)
11	12	13	14	15	16	17	18	19	20
(a)	(c)	(b)	(b)	(a)	(b)	(b)	(b)	(b)	(c)

Miscellaneous Problems

1. $AC = \dfrac{10}{x} - 4x^2 + 3x^3, MC = -12x^2 + 12x^3, MAC = \dfrac{-10}{x^2} - 8x + 9x^2$

2.(i) $n_d = \dfrac{-1}{x+1}$ (ii) $n_d = \dfrac{1}{x-1}$ (iii) $n_d = \dfrac{3}{x}$

3. $n_s = \dfrac{4p^2}{2p^2+5}, \dfrac{4}{7}$

7. Financial Mathematics

Exercise 7.1

1. ₹ 68,428.28 2. ₹ 1,20,800 3. ₹ 18,930 4. ₹ 500

5. ₹ 13,59,164 6. ₹ 14,736 7. ₹ 8,433 8. ₹ 1,17,612

9. ₹ 1,67,952 10. ₹ 1,000

Exercise 7.2

1. ₹ 8,184 2. ₹ 2,250 3. 900 shares

4.(i) 242 (ii) ₹ 3630 (iii) $12\frac{1}{2}\%$

5.(i) ₹ 4,000 (ii) ₹ 5,000 (iii) 9.6% 6. ₹ 8,975

7.(i) ₹ 6000 , (ii) ₹ 7500 8. 99 shares

9.(i) ₹ 945 (ii) ₹ 960 2nd investment is better.

10.(i) 1400 (ii) 1400. For the same investment both stocks fetch the same income. Therefore they are equivalent shares.

Exercise 7.3

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(a)	(b)	(c)	(c)	(d)	(c)	(b)	(b)	(b)	(c)	(b)	(a)	(c)	(c)	(c)

Miscellaneous Problems

1. ₹ 9280 2. ₹ 15644 3. ₹ 4328.57, ₹ 125800, ₹ 29340, ₹ 139600.

4. Required months= 24 5. ₹ 12500

6. ₹ 13240, ₹ 36420, Machine B may be purchased

7. ₹ 270, ₹ 216, ₹ 300, ₹ 450

8. ₹ 700, ₹ 900, ₹ 200, 2.5% 9. 500 shares, ₹ 625 10. 20%

8. Descriptive statistics and Probability

Exercise 8.1

1. $Q_1 = 6$, $Q_3 = 18$ 2. $Q_1 = 5$, $Q_3 = 6.5$, $D_8 = 6.5$ and $P_{67} = 6$

3. $Q_1 = 47.14$, $Q_3 = 63.44$, $D_5 = 55.58$, $D_7 = 61.56$ and $P_{60} = 58.37$

4. $GM= 142.5$ lbs 5. $GM= 26.2\%$

6. 192 km/hr 7. 38.92 km/hr

8. $AM=36$ $GM=25.46$ $HM=17.33$

9. $AM=21.96$ $GM=18.31$ $HM=14.32$

10. $AM=33$, $GM=29.51$, $HM=24.10$

11. $Q_1 = 30$, $Q_3 = 70$, $Q_D = 20$, Coefficient of $QD = 0.4$

12. $QD= 11.02$, Coefficient of $QD = 0.3384$

13. Median = 61, MD =1.71

14. Mean = 13, MD = 21.67

15. Median = 45.14 , MD = 14.30

Exercise 8.2

1. 1/3 2. 2/5

3. A and B are independent events 4.(i) 2/3 (ii) ½

5. 3/10 5.(i) 42/625 (ii) 207/625

7. 33/68 8.(i) 7/29 (ii) 5/29 (iii) 17/29

9. 4/11 10. P(A)=4/7 P(B) = 2/7 P(C) = 1/7

11.(i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ 12. $\frac{1}{2}$ 13. 0.2

14. 0.012 15.(i) 1/221 (ii) 1/7 16. P(C/D)= 0.5208

Exercise 8.3

1	2	3	4	5	6	7	8	9	10	11	12	13
(d)	(c)	(c)	(a)	(c)	(a)	(d)	(c)	(b)	(a)	(d)	(c)	(a)
14	15	16	17	18	19	20	21	22	23	24	25	
(b)	(d)	(a)	(b)	(b)	(c)	(b)	(d)	(b)	(a)	(b)	(a)	

Miscellaneous Problems

1. 16.02 tons 2. 16 3. Median=28, MD = 10.16

4. Mean=7.5, MD =2.3 5. QD=8.33, Coefficient of QD=0.21

6. 0.45 7.(i) 3/10 (ii) 3/5 (iii) 1/10

8. 0.948 9. 0.727 10. 0.493

9. Correlation and regression analysis

Exercise 9.1

1. 0.575 2. 0.94 3. 0.996 4. 0.891

5. 0.225 6. −0.0734 7. 0.9 8. 0.224

9. 0.905 10. −0.37

Exercise 9.2

1.(a) $Y=-0.66X+59.12$; $X=-0.234Y+40.892$

(b) $r=-0.394$ (c) $Y= 39.32$

2. $Y=0.6102X+66.12$; $X=0.556Y+74.62$ Height of son is 166.19

3. $Y=2.3X-35.67$; weight of the student is 125.79 lb

4. $Y=0.24X+1.04$; $X=1.33Y+1.34$

5. $Y=1.6X$; estimated yield = 46.4 unit per acre

6. $Y=0.942X+6.08$; Estimated sales =34.34 (in crores of rupees)

7. $Y=0.48X+67.72$; $X=0.91Y-41.35$; $Y=72.52$

8. $b_{xy}=0.33$; $b_{yx}=1.33$; $r=0.6667$

9. $Y=0.1565X+19.94$; estimated expenditure on food and entertainment (Y) is 51.24

10. $X=0.8Y-1$ and estimated value of X is 5.4 when $Y=8$

    $Y=0.8X+2.6$ and estimated value of Y is 12.2 when $X=12$

11. $\bar{X} = 13$; $\bar{Y} = 17$ and $r = 0.6$

12. $b_{xy} = -\frac{3}{2}; b_{yx} = -\frac{1}{2}; r = -0.866$

Exercise 9.3

1	2	3	4	5	6	7	8	9	10	11	12	13
(a)	(b)	(a)	(b)	(c)	(a)	(a)	(b)	(a)	(b)	(a)	(a)	(c)
14	15	16	17	18	19	20	21	22	23	24	25	
(a)	(b)	(a)	(a)	(a)	(a)	(a)	(b)	(b)	(b)	(a)	(d)	

Miscellaneous Problems

1. 0.906   2. 0.382   3. 0.95   4. 0.667

5. 0.905   6. Y=0.653X+21.71; Marks in B = 55.67

7. Y=0.576X+2.332; Y =5.788   8. Y=1.138X+80.78; X=0.706Y−46.742

9. $\bar{X}$ = 20, $\bar{Y}$ = 25, r = 0.8   10. b_{yx}=1.422, Y = 141.67

10. Operation research

Exercise: 10.1

1. Maximize $Z = 5x_1 + 3x_2$ subject to constraints $2x_1 + x_2 \leq 1000$; $x_1 \leq 400; x_2 \leq 700$ and $x_1, x_2 \geq 0$.

2. Maximize $Z = 30x_1 + 40x_2$ subject to constraints $60x_1 + 120x_2 \leq 12000$ $8x_1 + 5x_2 \leq 600; 3x_1 + 4x_2 \leq 500$ and x_1 , $x_2 \geq 0$.

3. Maximize $Z = 10x_1 + 150x_2$ subject to constraints $0.8x_1 + 1.2x_2 \leq 720; x_1 \leq 600$; $x_2 \leq 400$ and x_1, $x_2 \geq 0$.

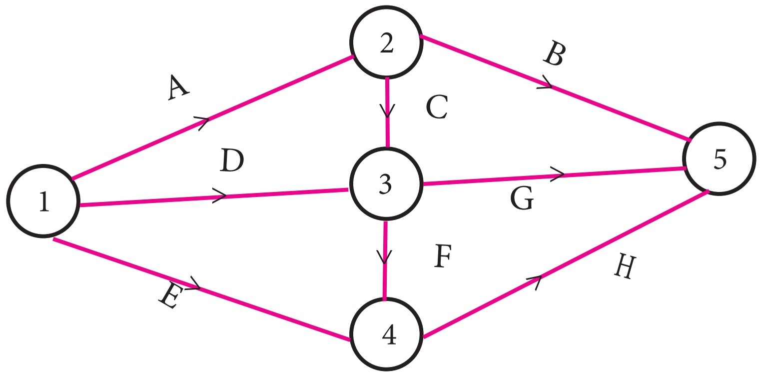
4.(i) $x_1 = 4$; $x_2 = 9$ and $Z_{max} = 96$ (ii) $x_1 = 8$; $x_2 = 12$ and $Z_{max} = 392$

(iii) $x_1 = 1$; $x_2 = 5$ and $Z_{min} = 13$ (iv) $x_1 = 2$; $x_2 = 3$ and $Z_{max} = 230$

(v) $x_1 = 3$; $x_2 = 9$ and $Z_{max} = 330$ (vi) $x_1 = 4$; $x_2 = 2$ and $Z_{min} = 160$

Exercise: 10.2

1.



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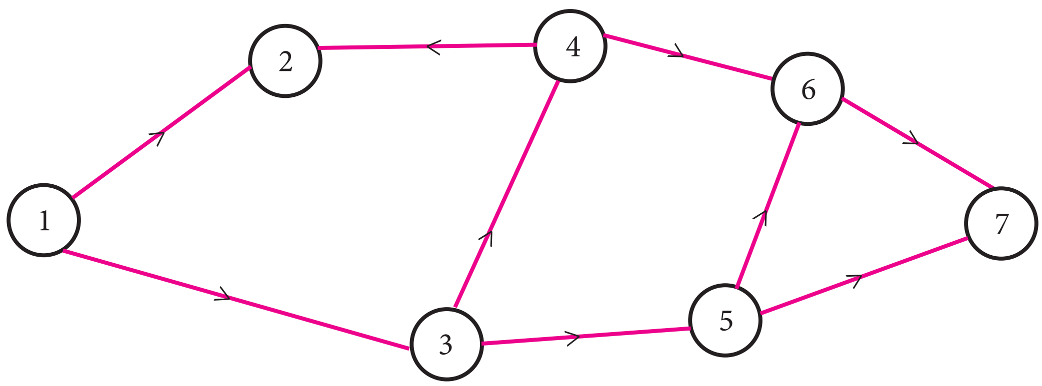

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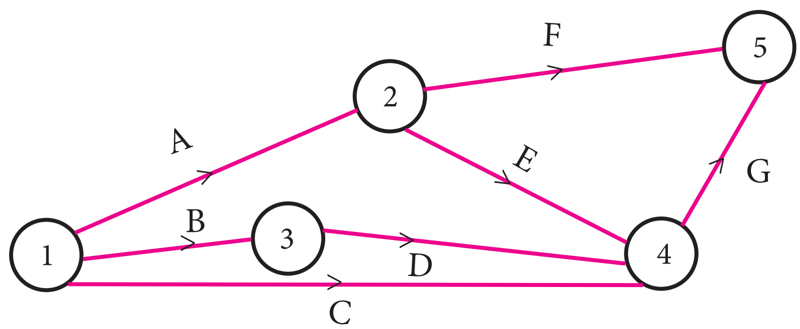

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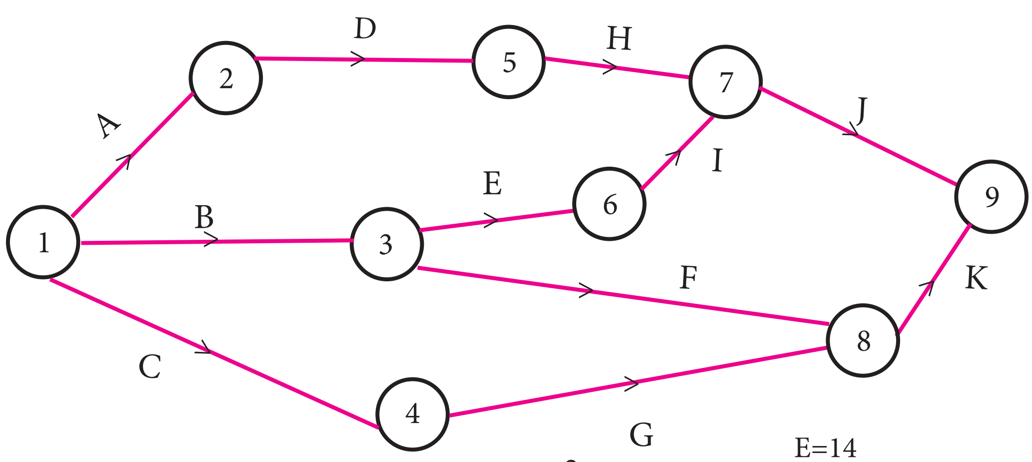
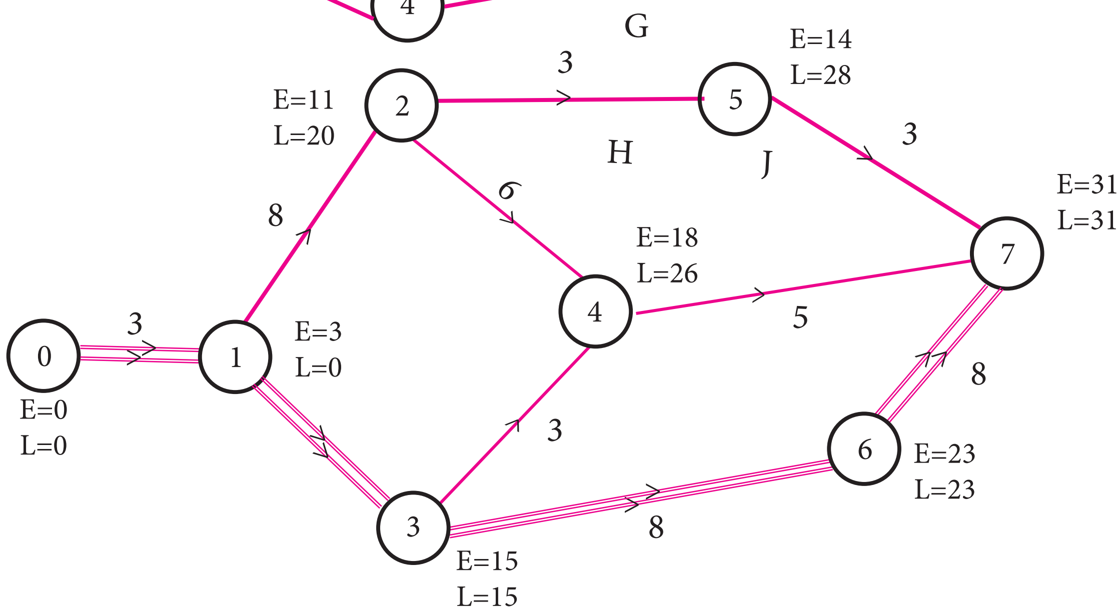
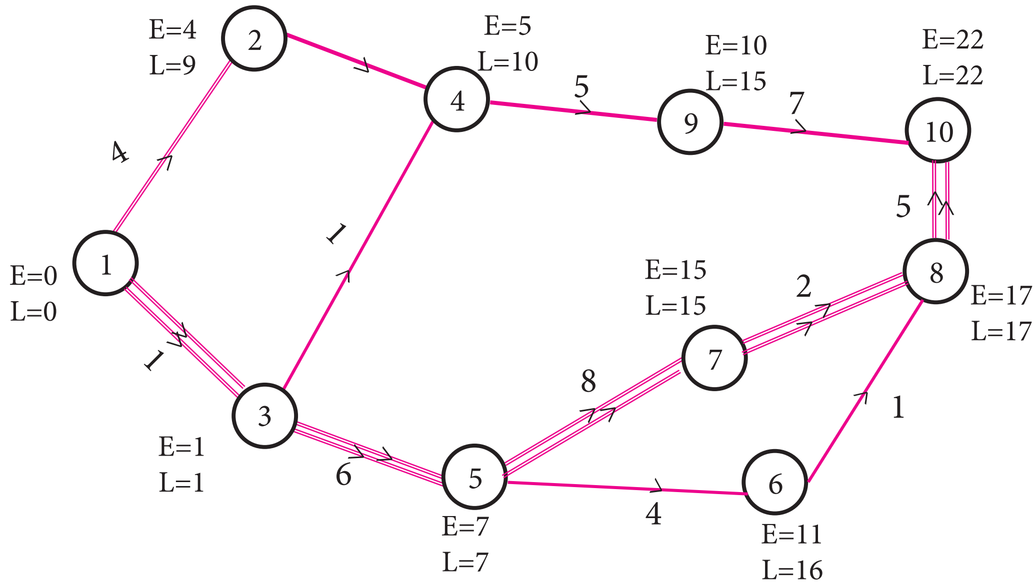


5.



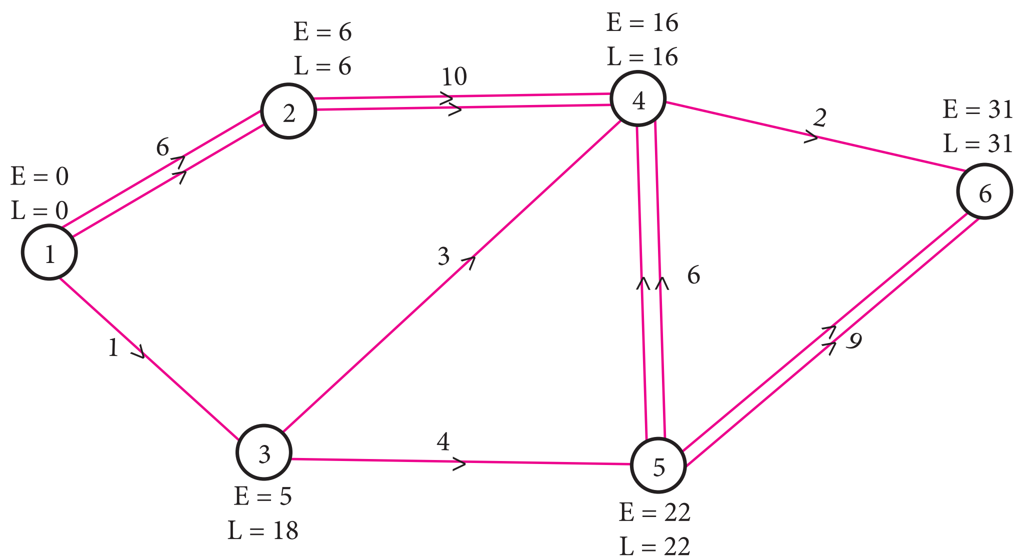
Critical path 0-1-3-6-7 and the duration is 31 weeks

6.



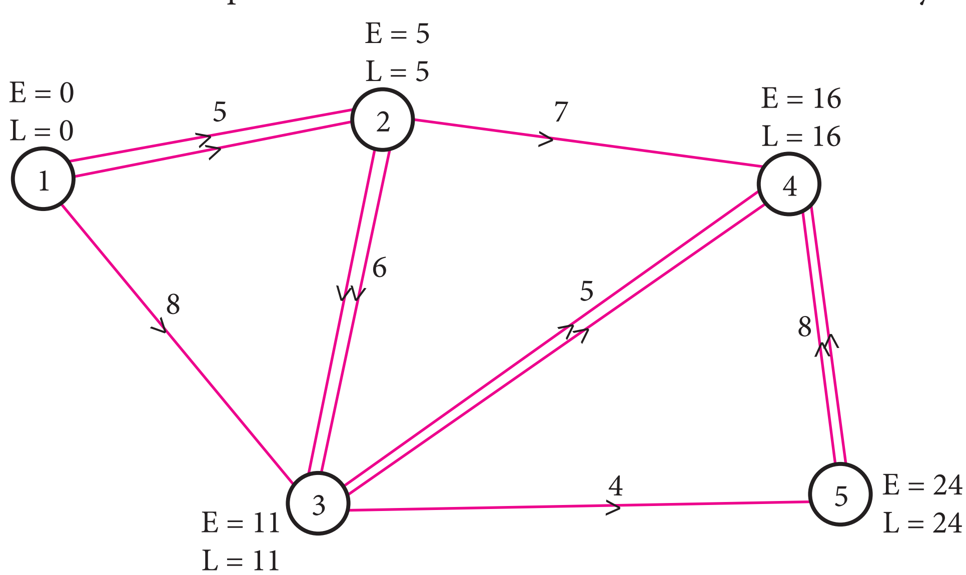
Critical path 1-3-5-7-8-10 and duration 22 time units

7.



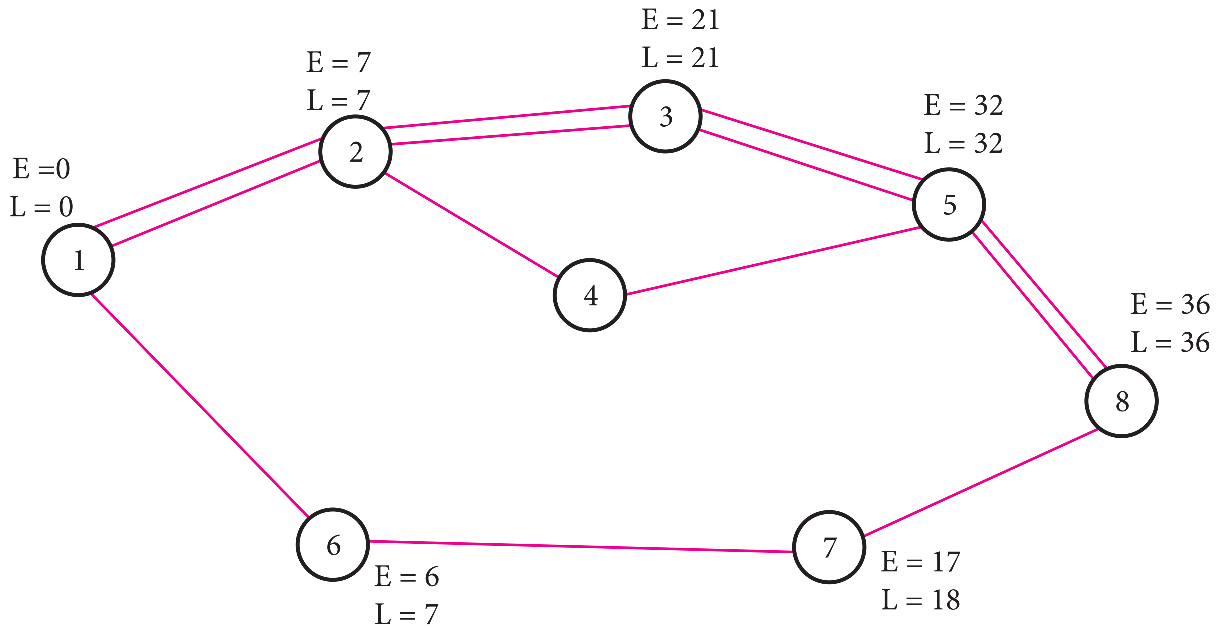
Critical path 1-2-4-5-6 and duration time taken is 31 days

8



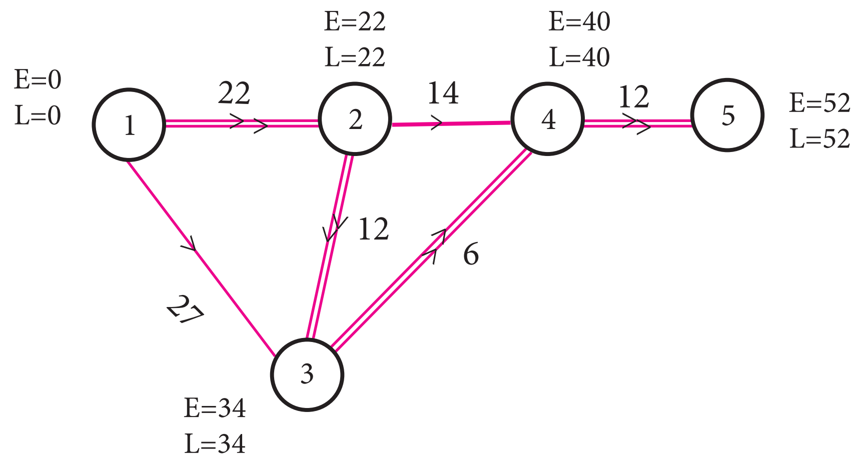
Critical path 1-2-3-4-5 and duration time taken is 24 days

9.



Critical path 1-2-3-5-8 and duration time taken is 36 days

10.



Critical path 1-2-3-4-5 and duration time taken is 52 days

Exercise-10.3

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(d)	(c)	(a)	(b)	(d)	(c)	(c)	(b)	(c)	(d)	(d)	(b)	(d)	(a)	(a)

Miscellaneous problems

1. Maximize $Z = 3x_1 + 4x_2$ subject to constraints $x_1 + x_2 \leq 450$; $2x_1 + x_2 \leq 600$ and $x_1, x_2 \geq 0$

2. Maximize $Z = x_1 + x_2$ subject to constraints $2x_1 + x_2 \geq 720$; $5x_1 + 8x_2 \geq 74$; $x_1 + 6x_2 \geq 24$ and $x_1, x_2 \geq 0$.

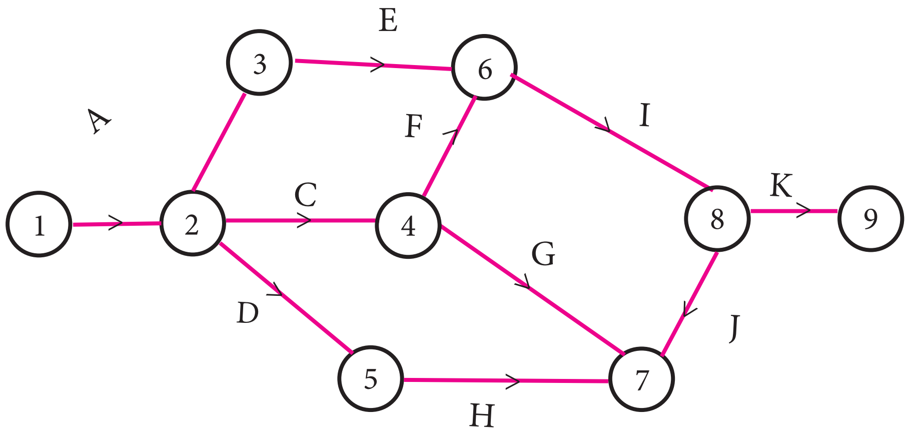
3. $x_1=30$; $x_2 = 0$ and $Z_{max} = 120$

4. $x_1 = 4$; $x_2 = 3$ and $Z_{min} = 2300$

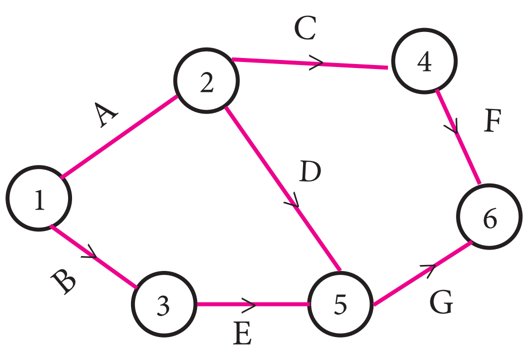
5. $x_1 = 1$; $x_2 = 5$ and Z_max= 28

6. $x_1 = 20$; $x_2=30$ and $Z_{max} = 1650$

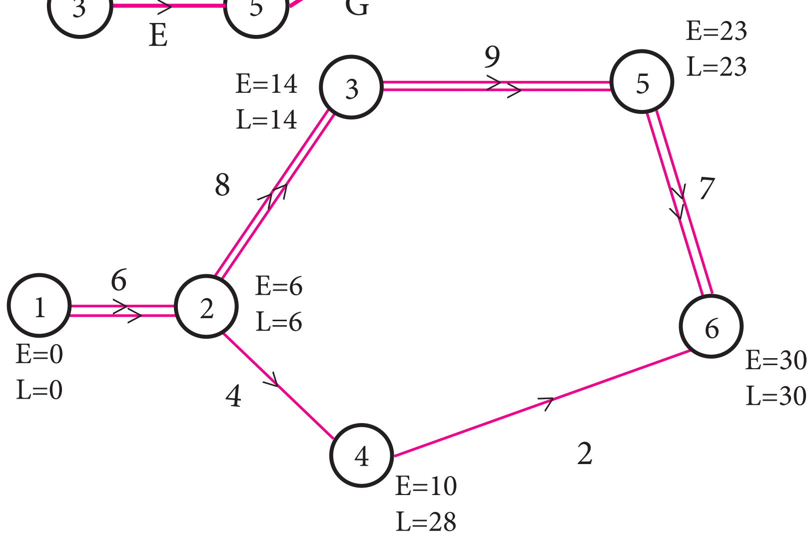
7.



8.

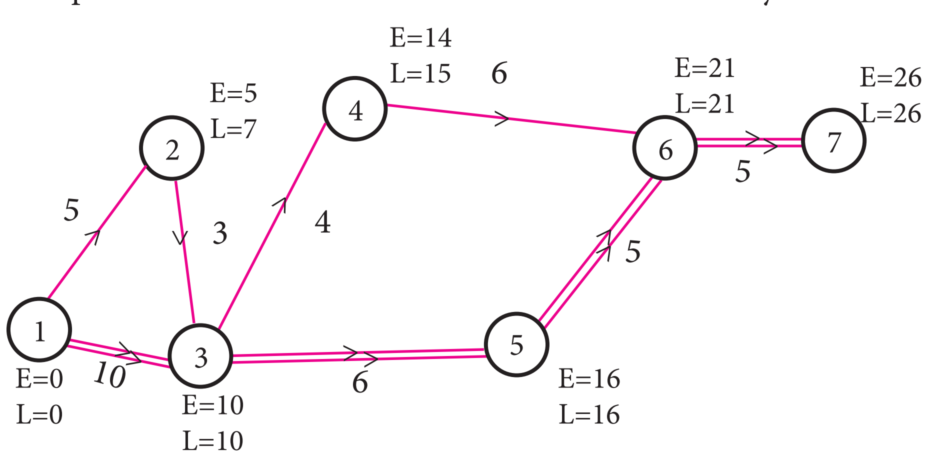


9.



Critical path 1-2-3-5-6 and duration time taken is 30 days.

10.



Critical path 1-3-5-6-7 and duration time taken is 26 days.

GLOSSARY	
Abnormal	அசாதாரனமான
abstract	பண்பு தொகை
activity	செயல்பாடு
Approximate	தோராயமாக
Approximately	தோரயமான
Assumed Mean	ஊகிக்கப்பட்ட சராசரி
Average	சராசரி
backward pass calculations	பின் நோக்கி செல்லும் கணக்கீடு
Bivariate analysis	இருமாறி பகுப்பாய்வு
Brokerage	தரகு
Capital value	மூலதன மதிப்பு
Characteristics	பண்புகள்
Closeness	பொருத்தமுடைய
Commodity	பொருள்
Conditional probability	நிபந்தனைக்குட்பட்ட நிகழ்தகவு
Consumer	நுகர்வோர்
Continuous series	தொடர்ச்சியான தொடர்
Corrleation	ஒட்டுறவு
Cost function	செலவுச் சார்பு
critical path analysis	தீர்வுக்கு உகந்த பகுப்பாய்வு
critical path method	தீர்வுக்கு உகந்த முறை
Debentures	கடன் பத்திரங்கள்
Decile	பதின்மானம்
decision variables	தீர்மான மாறிகள்
Demand	தேவை
Dependent events	சார்பு நிகழ்வுகள்
Deviations	விலக்கம்
Discrete series	தனித்த தொடர்
dummy activities	ஒப்புக்கான செயல்
earliest start time	முன்கூட்டியே தொடங்கும் நேரம்
Elasticity	நெகிழ்ச்சி
Equally likely events	சம வாய்ப்புள்ள நிகழ்வுகள்
Equilibrium	சமநிலை
Equity shares	சம பங்கு
event	நிகழ்வு
Excess	மிகுதியான
Exhaustive events	முழுமையான நிகழ்வுகள்
Face value	முக மதிப்பு
feasible solution	ஏற்புடைய தீர்வு
Fixed cost	ஒரே விலை/ மாறா விலை
fluctuate	ஏற்ற இறக்கம்
forward pass calculations	முன் நோக்கி செல்லும் கணக்கீடு
Frequency	அலைவெண் / நிகழ்வெண்
Grouped data	தொகுக்கப்பட்ட விவரங்கள்
head event	ஆரம்ப நிகழ்வு
Immediate annuity	தவணை பங்கீட்டு தொகை
Independent events	சார்பில்லா நிகழ்வுகள்
Interest	வட்டி
Interpretation	விளக்கம்

latest start time	சமீபத்திய தொடங்கும் நேரம்
linear programming problem	நேரிய திட்டமிடல் கணக்கு
logical sequence	தர்க்க தொடர் வரிசை
Marginal	இறுதிநிலை/ விளிம்பு
Market price	சந்தை விலை
Maximum	பெருமம்
Mean deviation	சராசரி விலக்கம்
Minimum	சிறுமம்
Mode	முகடு
Mutually exclusive events/ disjoint events	ஒன்றை ஒன்று விலக்கும் நிகழ்வுகள்
Negative Correlation	எதிர்மறை ஒட்டுறவு
network analysis	வலையமைப்பு பகுப்பாய்வு
Output	உற்பத்தி
Payment interval	செலுத்தும் கால இடைவெளி
Percentile	நூற்றுமானம்/ சதமானம்
Periodic payment	காலமுறை செலுத்துதல்
Perpetual annuity	நிரந்தர தவணை பங்கீட்டு தொகை
Positive Correlation	நேரிடை ஒட்டுறவு
predecessor activity	முந்தைய செயல்
Preference shares	முன்னுரிமை
Probability	நிகழ்தகவு
Producer	உற்பத்தியாளர்
Profit	இலாபம்
Quantity	அளவு
Quartile	கால்மானம்
Quartile deviation	கால்மான விலக்கம்
Random experiment	சமவாய்ப்பு சோதனை
Random variables	சமவாய்ப்பு மாறிகள்
Range	வீச்சு
Rate of change	மாறுவீதம்
Ratio	விகிதம்
Regression analysis	தொடர்பு போக்கு ஆய்வு
Relative change	சார்ந்த மாற்றம்
Relative Variable	சார்ந்த மாறி
Revenue function	வருவாய் சார்பு
Sample space	கூறுவெளி
Selling price	விற்பனை விலை
Share holders	பங்குதாரர்கள்
Shares	பங்குகள்
Stock exchange	பங்குச் சந்தை
Stocks	சரக்கு முதல்கள்
successor activity	பிந்தைய செயல்
Supply	அளிப்பு
tail event	இறுதி நிகழ்வு
Term of annuity	தவணை பங்கீட்டு தொகை காலம்
Transaction	பரிவர்த்தனை
Univariate analysis	ஒருமாறி பகுப்பாய்வு
Variable cost	மாறும் விலை

LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	3	7	10	14	17	21	24	28	31
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430	3	6	10	13	16	19	23	26	29
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	3	6	9	12	15	18	21	24	27
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014	3	6	8	11	14	17	20	22	25
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279	3	5	8	11	13	16	18	21	24
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529	2	5	7	10	12	15	17	20	22
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765	2	5	7	9	12	14	16	19	21
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989	2	4	7	9	11	13	16	18	20
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201	2	4	6	8	11	13	15	17	19
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598	2	4	6	8	10	12	14	15	17
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	2	4	6	7	9	11	13	15	17
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	2	4	5	7	9	11	12	14	16
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	2	3	5	7	9	10	12	14	15
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298	2	3	5	7	8	10	11	13	15
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456	2	3	5	6	8	9	11	13	14
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609	2	3	5	6	8	9	11	12	14
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757	1	3	4	6	7	9	10	12	13
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900	1	3	4	6	7	9	10	11	13
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038	1	3	4	6	7	8	10	11	12
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172	1	3	4	5	7	8	9	11	12
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302	1	3	4	5	6	8	9	10	12
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428	1	3	4	5	6	8	9	10	11
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551	1	2	4	5	6	7	9	10	11
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670	1	2	4	5	6	7	8	10	11
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786	1	2	3	5	6	7	8	9	10
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899	1	2	3	5	6	7	8	9	10
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010	1	2	3	4	5	7	8	9	10
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117	1	2	3	4	5	6	8	9	10
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222	1	2	3	4	5	6	7	8	9
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325	1	2	3	4	5	6	7	8	9
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425	1	2	3	4	5	6	7	8	9
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522	1	2	3	4	5	6	7	8	9
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618	1	2	3	4	5	6	7	8	9
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712	1	2	3	4	5	6	7	7	8
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803	1	2	3	4	5	5	6	7	8
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893	1	2	3	4	4	5	6	7	8
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981	1	2	3	4	4	5	6	7	8
5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067	1	2	3	3	4	5	6	7	8
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152	1	2	3	3	4	5	6	7	8
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235	1	2	2	3	4	5	6	7	7
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316	1	2	2	3	4	5	6	6	7
5.4	0.7324	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396	1	2	2	3	4	5	6	6	7

LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474	1	2	2	3	4	5	5	6	7
5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551	1	2	2	3	4	5	5	6	7
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627	1	2	2	3	4	5	5	6	7
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701	1	1	2	3	4	4	5	6	7
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774	1	1	2	3	4	4	5	6	7
6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846	1	1	2	3	4	4	5	6	6
6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917	1	1	2	3	4	4	5	6	6
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987	1	1	2	3	3	4	5	6	6
6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055	1	1	2	3	3	4	5	5	6
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122	1	1	2	3	3	4	5	5	6
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189	1	1	2	3	3	4	5	5	6
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254	1	1	2	3	3	4	5	5	6
6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319	1	1	2	3	3	4	5	5	6
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382	1	1	2	3	3	4	4	5	6
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445	1	1	2	2	3	4	4	5	6
7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506	1	1	2	2	3	4	4	5	6
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567	1	1	2	2	3	4	4	5	5
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627	1	1	2	2	3	4	4	5	5
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686	1	1	2	2	3	4	4	5	5
7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745	1	1	2	2	3	4	4	5	5
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802	1	1	2	2	3	3	4	5	5
7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859	1	1	2	2	3	3	4	5	5
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915	1	1	2	2	3	3	4	4	5
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971	1	1	2	2	3	3	4	4	5
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025	1	1	2	2	3	3	4	4	5
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079	1	1	2	2	3	3	4	4	5
8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133	1	1	2	2	3	3	4	4	5
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186	1	1	2	2	3	3	4	4	5
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238	1	1	2	2	3	3	4	4	5
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289	1	1	2	2	3	3	4	4	5
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340	1	1	2	2	3	3	4	4	5
8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390	1	1	2	2	3	3	4	4	5
8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440	0	1	1	2	2	3	3	4	4
8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489	0	1	1	2	2	3	3	4	4
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538	0	1	1	2	2	3	3	4	4
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586	0	1	1	2	2	3	3	4	4
9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633	0	1	1	2	2	3	3	4	4
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680	0	1	1	2	2	3	3	4	4
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727	0	1	1	2	2	3	3	4	4
9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773	0	1	1	2	2	3	3	4	4
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818	0	1	1	2	2	3	3	4	4
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863	0	1	1	2	2	3	3	4	4
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	0.9908	0	1	1	2	2	3	3	4	4
9.8	0.9912	0.9917	0.9921	0.9926	0.9930	0.9934	0.9939	0.9943	0.9948	0.9952	0	1	1	2	2	3	3	4	4
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996	0	1	1	2	2	3	3	3	4

ANTI LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.00	1.000	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
0.01	1.023	1.026	1.028	1.030	1.033	1.035	1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
0.02	1.047	1.050	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
0.03	1.072	1.074	1.076	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
0.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
0.05	1.122	1.125	1.127	1.130	1.132	1.135	1.138	1.140	1.143	1.146	0	1	1	1	1	2	2	2	2
0.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
0.07	1.175	1.178	1.180	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
0.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
0.09	1.230	1.233	1.236	1.239	1.242	1.245	1.247	1.250	1.253	1.256	0	1	1	1	1	2	2	2	3
0.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
0.11	1.288	1.291	1.294	1.297	1.300	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
0.12	1.318	1.321	1.324	1.327	1.330	1.334	1.337	1.340	1.343	1.346	0	1	1	1	2	2	2	2	3
0.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
0.14	1.380	1.384	1.387	1.390	1.393	1.396	1.400	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
0.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
0.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
0.17	1.479	1.483	1.486	1.489	1.493	1.496	1.500	1.503	1.507	1.510	0	1	1	1	2	2	2	3	3
0.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
0.19	1.549	1.552	1.556	1.560	1.563	1.567	1.570	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
0.20	1.585	1.589	1.592	1.596	1.600	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
0.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
0.22	1.660	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.690	1.694	0	1	1	2	2	2	3	3	3
0.23	1.698	1.702	1.706	1.710	1.714	1.718	1.722	1.726	1.730	1.734	0	1	1	2	2	2	3	3	4
0.24	1.738	1.742	1.746	1.750	1.754	1.758	1.762	1.766	1.770	1.774	0	1	1	2	2	2	3	3	4
0.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
0.26	1.820	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
0.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
0.28	1.905	1.910	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
0.29	1.950	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
0.30	1.995	2.000	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
0.31	2.042	2.046	2.051	2.056	2.061	2.065	2.070	2.075	2.080	2.084	0	1	1	2	2	3	3	4	4
0.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
0.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
0.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
0.35	2.239	2.244	2.249	2.254	2.259	2.265	2.270	2.275	2.280	2.286	1	1	2	2	3	3	4	4	5
0.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	1	2	2	3	3	4	4	5
0.37	2.344	2.350	2.355	2.360	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
0.38	2.399	2.404	2.410	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
0.39	2.455	2.460	2.466	2.472	2.477	2.483	2.489	2.495	2.500	2.506	1	1	2	2	3	3	4	5	5
0.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
0.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
0.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6

ANTI LOGARITHM TABLE

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
0.50	3.162	3.170	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1	1	2	3	4	4	5	6	7
0.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
0.52	3.311	3.319	3.327	3.334	3.342	3.350	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
0.53	3.388	3.396	3.404	3.412	3.420	3.428	3.436	3.443	3.451	3.459	1	2	2	3	4	5	6	6	7
0.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.540	1	2	2	3	4	5	6	6	7
0.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
0.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.690	3.698	3.707	1	2	3	3	4	5	6	7	8
0.57	3.715	3.724	3.733	3.741	3.750	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
0.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
0.59	3.890	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
0.60	3.981	3.990	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
0.61	4.074	4.083	4.093	4.102	4.111	4.121	4.130	4.140	4.150	4.159	1	2	3	4	5	6	7	8	9
0.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
0.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
0.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.550	4.560	1	2	3	4	5	6	7	8	9
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
0.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	8	9	10
0.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	6	7	8	9	10
0.69	4.898	4.909	4.920	4.932	4.943	4.955	4.966	4.977	4.989	5.000	1	2	3	5	6	7	8	9	10
0.70	5.012	5.023	5.035	5.047	5.058	5.070	5.082	5.093	5.105	5.117	1	2	4	5	6	7	8	9	11
0.71	5.129	5.140	5.152	5.164	5.176	5.188	5.200	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
0.72	5.248	5.260	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
0.73	5.370	5.383	5.395	5.408	5.420	5.433	5.445	5.458	5.470	5.483	1	3	4	5	6	8	9	10	11
0.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.610	1	3	4	5	6	8	9	10	12
0.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
0.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
0.77	5.888	5.902	5.916	5.929	5.943	5.957	5.970	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
0.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
0.79	6.166	6.180	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.310	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.730	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
0.84	6.918	6.934	6.950	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
0.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2	3	5	7	8	10	12	13	15
0.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
0.87	7.413	7.430	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
0.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	4	5	7	9	11	12	14	16
0.89	7.762	7.780	7.798	7.816	7.834	7.852	7.870	7.889	7.907	7.925	2	4	5	7	9	11	13	14	16
0.90	7.943	7.962	7.980	7.998	8.017	8.035	8.054	8.072	8.091	8.110	2	4	6	7	9	11	13	15	17
0.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.260	8.279	8.299	2	4	6	8	9	11	13	15	17
0.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
0.93	8.511	8.531	8.551	8.570	8.590	8.610	8.630	8.650	8.670	8.690	2	4	6	8	10	12	14	16	18
0.94	8.710	8.730	8.750	8.770	8.790	8.810	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
0.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
0.96	9.120	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.290	9.311	2	4	6	8	11	13	15	17	19
0.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
0.98	9.550	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.750	2	4	7	9	11	13	16	18	20
0.99	9.772	9.795	9.817	9.840	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	20

EXPONENTIAL FUNCTION TABLE

	0	1	2	3	4	5	6	7	8	9
0.00	1.00000000	2.71828183	7.38905610	20.08553692	54.59815003	148.41315910	403.42879349	1096.63315843	2980.95798704	8103.08392758
0.01	1.01005017	2.74560102	7.46331735	20.28739993	55.14687056	149.90473615	407.48332027	1107.65450490	3010.91711288	8184.52127494
0.02	1.02020134	2.77319476	7.53832493	20.49129168	55.70110583	151.41130379	411.57859573	1118.78661775	3041.17733294	8266.77708126
0.03	1.03045453	2.80106583	7.61408636	20.69723259	56.26091125	152.93301270	415.71502938	1130.03061019	3071.74167327	8349.85957218
0.04	1.04081077	2.82921701	7.69060920	20.90524324	56.82634281	154.47001503	419.89303489	1141.38760663	3102.61319033	8433.77705601
0.05	1.05127110	2.85765112	7.76790111	21.11534442	57.39745705	156.02246449	424.11303004	1152.85874278	3133.79497129	8518.53792457
0.06	1.06183655	2.88637099	7.84596981	21.32755716	57.97431108	157.59051632	428.37543686	1164.44516577	3165.29013436	8604.15065402
0.07	1.07250818	2.91537950	7.92482312	21.54190268	58.55696259	159.17432734	432.68068157	1176.14803425	3197.10182908	8690.62380571
0.08	1.08328707	2.94467955	8.00446891	21.75840240	59.14546985	160.77405593	437.02919472	1187.96851851	3229.23323664	8777.96602703
0.09	1.09417428	2.97427407	8.08491516	21.97707798	59.73989170	162.38986205	441.42141115	1199.90780061	3261.68757023	8866.18605226
0.10	1.10517092	3.00416602	8.16616991	22.19795128	60.34028760	164.02190730	445.85777008	1211.96707449	3294.46807528	8955.29270348
0.11	1.11627807	3.03435839	8.24824128	22.42104440	60.94671757	165.67035487	450.33871517	1224.14754609	3327.57802989	9045.29489144
0.12	1.12749685	3.06485420	8.33113749	22.64637964	61.55924226	167.33536962	454.86469450	1236.45043347	3361.02074508	9136.20161642
0.13	1.13882838	3.09565650	8.41486681	22.87397954	62.17792293	169.01711804	459.43616068	1248.87696691	3394.79956514	9228.02196918
0.14	1.15027380	3.12676837	8.49943763	23.10386686	62.80282145	170.71576832	464.05357086	1261.42838910	3428.91786799	9320.76513183
0.15	1.16183424	3.15819291	8.58485840	23.33606458	63.43400030	172.43149032	468.71738678	1274.10595517	3463.37906548	9414.44037876
0.16	1.17351087	3.18993328	8.67113766	23.57059593	64.07152260	174.16445561	473.42807483	1286.91093291	3498.18660376	9509.05707757
0.17	1.18530485	3.22199264	8.75828404	23.80748436	64.71545211	175.91483748	478.18610609	1299.84460280	3533.34396362	9604.62469001
0.18	1.19721736	3.25437420	8.84630626	24.04675355	65.36585321	177.68281099	482.99195635	1312.90825825	3568.85466082	9701.15277293
0.19	1.20924960	3.28708121	8.93521311	24.28842744	66.02279096	179.46855293	487.84610621	1326.10320561	3604.72224646	9798.65097920
0.20	1.22140276	3.32011692	9.02501350	24.53253020	66.68633104	181.27224188	492.74904109	1339.43076439	3640.95030733	9897.12905874
0.21	1.23367806	3.35348465	9.11571639	24.77908622	67.35653981	183.09405819	497.70125129	1352.89226737	3677.54246627	9996.59685944
0.22	1.24607673	3.38718773	9.20733087	25.02812018	68.03348429	184.93418407	502.70323202	1366.48906071	3714.50238251	10097.06432815
0.23	1.25860001	3.42122954	9.29986608	25.27965697	68.71723217	186.79280352	507.75548350	1380.22250409	3751.83375209	10198.54151171
0.24	1.27124915	3.45561346	9.39333129	25.53372175	69.40785184	188.67010241	512.85851094	1394.09397087	3789.54030817	10301.03855791
0.25	1.28402542	3.49034296	9.48773584	25.79033992	70.10541235	190.56626846	518.01282467	1408.10484820	3827.62582144	10404.56571656
0.26	1.29693009	3.52542149	9.58308917	26.04953714	70.80998345	192.48149130	523.21894011	1422.25653720	3866.09410048	10509.13334045
0.27	1.30996445	3.56085256	9.67940081	26.31133934	71.52163562	194.41596245	528.47737788	1436.55045304	3904.94899215	10614.75188643
0.28	1.32312981	3.59663973	9.77668041	26.57577270	72.24044001	196.36987535	533.78866383	1450.98802511	3944.19438198	10721.43191645
0.29	1.33642749	3.63278656	9.87493768	26.84286366	72.96646850	198.34342541	539.15332908	1465.57069720	3983.83419453	10829.18409859
0.30	1.34985881	3.66929667	9.97418245	27.11263892	73.69979370	200.33680997	544.57191013	1480.29992758	4023.87239382	10938.01920817
0.31	1.36342511	3.70617371	10.07442466	27.38512547	74.44048894	202.35022839	550.04494881	1495.17718919	4064.31298371	11047.94812878
0.32	1.37712776	3.74342138	10.17567431	27.66035056	75.18862829	204.38388199	555.57299245	1510.20396976	4105.16000827	11158.98185341
0.33	1.39096813	3.78104339	10.27794153	27.93834170	75.94428657	206.43797416	561.15659385	1525.38177199	4146.41755226	11271.13148552
0.34	1.40494759	3.81904351	10.38123656	28.21912671	76.70753934	208.51271029	566.79631138	1540.71211367	4188.08974147	11384.40824018
0.35	1.41906755	3.85742553	10.48556972	28.50273364	77.47846293	210.60829787	572.49270901	1556.19652784	4230.18074313	11498.82344515
0.36	1.43332941	3.89619330	10.59095145	28.78919088	78.25713442	212.72494645	578.24635639	1571.83656296	4272.69476640	11614.38854204
0.37	1.44773461	3.93535070	10.69739228	29.07852706	79.04363170	214.86286770	584.05782889	1587.63378304	4315.63606270	11731.11508747
0.38	1.46228459	3.97490163	10.80490286	29.37077111	79.83803341	217.02227542	589.92770766	1603.58976783	4359.00892620	11849.01475419
0.39	1.47698079	4.01485005	10.91349394	29.66595227	80.64041898	219.20338555	595.85657969	1619.70611293	4402.81769423	11968.09933225
0.40	1.49182470	4.05519997	11.02317638	29.96410005	81.45086866	221.40641620	601.84503787	1635.98443000	4447.06674770	12088.38073022
0.41	1.50681779	4.09595540	11.13396115	30.26524426	82.26946350	223.63158768	607.89368106	1652.42634686	4491.76051155	12209.87097633
0.42	1.52196156	4.13712044	11.24585931	30.56941502	83.09628536	225.87912250	614.00311413	1669.03350774	4536.90345519	12332.58221972
0.43	1.53725752	4.17869919	11.35888208	30.87664275	83.93141691	228.14924542	620.17394801	1685.80757337	4582.50009296	12456.52673161
0.44	1.55270722	4.22069582	11.47304074	31.18695817	84.77494167	230.44218346	626.40679981	1702.75022115	4628.55498456	12581.71690655
0.45	1.56831219	4.26311452	11.58834672	31.50039231	85.62694400	232.75816591	632.70229281	1719.86314538	4675.07273551	12708.16526367
0.46	1.58407398	4.30595953	11.70481154	31.81697651	86.48750910	235.09742437	639.06105657	1737.14805735	4722.05799763	12835.88444790
0.47	1.59999419	4.34923514	11.82244685	32.13674244	87.35672301	237.46019276	645.48372697	1754.60668558	4769.51546949	12964.88723127
0.48	1.61607440	4.39294568	11.94126442	32.45972208	88.23467268	239.84670737	651.97094627	1772.24077593	4817.44989687	13095.18651418
0.49	1.63231622	4.43709552	12.06127612	32.78594771	89.12144588	242.25720686	658.52336322	1790.05209184	4865.86607325	13226.79532664
0.50	1.64872127	4.48168907	12.18249396	33.11545196	90.01713130	244.69193226	665.14163304	1808.04241446	4914.76884030	13359.72682966

EXPONENTIAL FUNCTION TABLE

	0	1	2	3	4	5	6	7	8	9
0.51	1.66529119	4.52673079	12.30493006	33.44826778	90.92181851	247.15112707	671.82641759	1826.21354282	4964.16308832	13493.99431650
0.52	1.68202765	4.57222520	12.42859666	33.78442846	91.83559798	249.63503719	678.57838534	1844.56729405	5014.05375679	13629.61121401
0.53	1.69893231	4.61817682	12.55350614	34.12396761	92.75856108	252.14391102	685.39821149	1863.10550356	5064.44583482	13766.59108401
0.54	1.71600686	4.66459027	12.67967097	34.46691919	93.69080012	254.67799946	692.28657804	1881.83002516	5115.34436165	13904.94762458
0.55	1.73325302	4.71147018	12.80710378	34.81331749	94.63240831	257.23755591	699.24417382	1900.74273134	5166.75442718	14044.69467150
0.56	1.75067250	4.75882125	12.93581732	35.16319715	95.58347983	259.82283632	706.27169460	1919.84551337	5218.68117245	14185.84619960
0.57	1.76826705	4.80664819	13.06582444	35.51659315	96.54410977	262.43409924	713.36984313	1939.14028156	5271.12979019	14328.41632413
0.58	1.78603843	4.85495581	13.19713816	35.87354085	97.51439421	265.07160579	720.53932925	1958.62896539	5324.10552531	14472.41930224
0.59	1.80398842	4.90374893	13.32977160	36.23407593	98.49443016	267.73561971	727.78086990	1978.31351375	5377.61367541	14617.86953434
0.60	1.82211880	4.95303242	13.46373804	36.59823444	99.48431564	270.42640743	735.09518924	1998.19589510	5431.65959136	14764.78156558
0.61	1.84043140	5.00281123	13.59905085	36.96605281	100.48414964	273.14423800	742.48301872	2018.27809772	5486.24867780	14913.17008727
0.62	1.85892804	5.05309032	13.73572359	37.33756782	101.49403213	275.88938323	749.94509711	2038.56212982	5541.38639368	15063.04993840
0.63	1.87761058	5.10387472	13.87376990	37.71281662	102.51406411	278.66211763	757.48217064	2059.05001984	5597.07825281	15214.43610708
0.64	1.89648088	5.15516951	14.01320361	38.09183673	103.54434758	281.46271848	765.09499302	2079.74381657	5653.32982444	15367.34373205
0.65	1.91554083	5.20697983	14.15403865	38.47466605	104.58498558	284.29146582	772.78432554	2100.64558942	5710.14673375	15521.78810420
0.66	1.93479233	5.25931084	14.29628910	38.86134287	105.63608216	287.14864256	780.55093713	2121.75742858	5767.53466250	15677.78466809
0.67	1.95423732	5.31216780	14.43996919	39.25190586	106.69774243	290.03453439	788.39560446	2143.08144525	5825.49934952	15835.34902351
0.68	1.97387773	5.36555597	14.58509330	39.64639407	107.77007257	292.94942992	796.31911202	2164.61977185	5884.04659134	15994.49692704
0.69	1.99371553	5.41948071	14.73167592	40.04484696	108.85317981	295.89362064	804.32225214	2186.37456223	5943.18224271	16155.24429358
0.70	2.01375271	5.47394739	14.87973172	40.44730436	109.94717245	298.86740097	812.40582517	2208.34799189	6002.91221726	16317.60719802
0.71	2.03399126	5.52896148	15.02927551	40.85380653	111.05215991	301.87106828	820.57063945	2230.54225819	6063.24248804	16481.60187677
0.72	2.05443321	5.58452846	15.18032224	41.26439411	112.16825267	304.90492296	828.81751148	2252.95958057	6124.17908811	16647.24472945
0.73	2.07508061	5.64065391	15.33288702	41.67910816	113.29556235	307.96926838	837.14726595	2275.60220079	6185.72811120	16814.55232047
0.74	2.09593551	5.69734342	15.48698510	42.09799016	114.43420168	311.06441098	845.56073585	2298.47238312	6247.89571226	16983.54138073
0.75	2.11700002	5.75460268	15.64263188	42.52108200	115.58428453	314.19066029	854.05876253	2321.57241461	6310.68810809	17154.22880929
0.76	2.13827622	5.81243739	15.79984295	42.94842598	116.74592590	317.34832892	862.64219579	2344.90460528	6374.11157799	17326.63167502
0.77	2.15976625	5.87085336	15.95863401	43.38006484	117.91924196	320.53773265	871.31189399	2368.47128836	6438.17246436	17500.76721836
0.78	2.18147227	5.92985642	16.11902095	43.81604174	119.10435004	323.75919042	880.06872411	2392.27482054	6502.87717335	17676.65285301
0.79	2.20339643	5.98945247	16.28101980	44.25640028	120.30136866	327.01302438	888.91356183	2416.31758219	6568.23217547	17854.30616767
0.80	2.22554093	6.04964746	16.44464677	44.70118449	121.51041752	330.29955991	897.84729165	2440.60197762	6634.24400628	18033.74492783
0.81	2.24790799	6.11044743	16.60991822	45.15043887	122.73161752	333.61912567	906.87080695	2465.13043529	6700.91926702	18214.98707751
0.82	2.27049984	6.17185845	16.77685067	45.60420832	123.96509078	336.97205363	915.98501008	2489.90540804	6768.26462527	18398.05074107
0.83	2.29331874	6.23388666	16.94546082	46.06253823	125.21096065	340.35867907	925.19081248	2514.92937342	6836.28681562	18582.95422504
0.84	2.31636698	6.29653826	17.11576554	46.52547444	126.46935173	343.77934066	934.48913473	2540.20483383	6904.99264036	18769.71601992
0.85	2.33964685	6.35981952	17.28778184	46.99306323	127.74038985	347.23438048	943.88090667	2565.73431683	6974.38897011	18958.35480204
0.86	2.36316069	6.42373677	17.46152694	47.46535137	129.02420211	350.72414402	953.36706749	2591.52037541	7044.48274457	19148.88943544
0.87	2.38691085	6.48829640	17.63701820	47.94238608	130.32091690	354.24898027	962.94856581	2617.56558819	7115.28097317	19341.33897375
0.88	2.41089971	6.55350486	17.81427318	48.42421507	131.63066389	357.80924171	972.62635979	2643.87255970	7186.79073580	19535.72266207
0.89	2.43512965	6.61936868	17.99330960	48.91088652	132.95357405	361.40528437	982.40141722	2670.44392068	7259.01918349	19732.05993893
0.90	2.45960311	6.68589444	18.17414537	49.40244911	134.28977968	365.03746787	992.27471561	2697.28232827	7331.97353916	19930.37043823
0.91	2.48432253	6.75308880	18.35679857	49.89895197	135.63941441	368.70615541	1002.24724229	2724.39046634	7405.66109828	20130.67399118
0.92	2.50929039	6.82095847	18.54128746	50.40044478	137.00261319	372.41171388	1012.31999453	2751.77104573	7480.08922969	20332.99062831
0.93	2.53450918	6.88951024	18.72763050	50.90697767	138.37951234	376.15451382	1022.49397962	2779.42680452	7555.26537625	20537.34058145
0.94	2.55998142	6.95875097	18.91584631	51.41860130	139.77024956	379.93492954	1032.77021496	2807.36050830	7631.19705565	20743.74428576
0.95	2.58570966	7.02868758	19.10595373	51.93536683	141.17496392	383.75333906	1043.14972818	2835.57495047	7707.89186111	20952.22238178
0.96	2.61169647	7.09932707	19.29797176	52.45732595	142.59379590	387.61012424	1053.63355724	2864.07295251	7785.35746218	21162.79571750
0.97	2.63794446	7.17067649	19.49191960	52.98453084	144.02688737	391.50567075	1064.22275054	2892.85736422	7863.60160548	21375.48535043
0.98	2.66445624	7.24274299	19.68781664	53.51703423	145.47438165	395.44036816	1074.91836700	2921.93106408	7942.63211550	21590.31254971
0.99	2.69123447	7.31553376	19.88568249	54.05488936	146.93642350	399.41460993	1085.72147619	2951.29695948	8022.45689535	21807.29879823

Books for Reference

1. Introduction to Matrices, S.P.Gupta , S.Chand & Company
2. Matrices, Shanthi Narayanan,S.Chand & Company
3. Matrices and Determinants,P.N.Arora, S.Chand & Company
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32. Mathematics, HSC First & Second Year, Tamilnadu Text Book Corporation, Reprint 2017

Books for Reference

1. Introduction to Matrices, S.P.Gupta , S.Chand & Company
2. Matrices, Shanthi Narayanan,S.Chand & Company
3. Matrices and Determinants,P.N.Arora, S.Chand & Company
4. Topics in Algebra, I.N.Herstein, Vikas Publishing Company
5. Algebra A Complete Course , R.D.Sharma, Sultan Chand & Sons
6. Analytical Geometry, T.K.Manicavachagon Pillay, S.Narayanan, S.Viswanathan Publishers
7. Analytical Geometry,P.K.Mittal, Shanthi Narayanan, Durai Pandiyan, S.Chand & Company
8. Trigonometry, R.D.Sharma, Sulatan Chand & Sons
9. A Text Book of Trigonometry, M.D Raisingania and Aggarwal
10. Trigonometry, D.C.Sharma, V.K.Kapoor, Sulatan Chand & Sons
11. Trignonometry, S.Arumugam , S.Narayanan, T.K.Manicavachagon Pillay, New Gama Publications, S.Viswanathan Printers and Publishers Pvt. Ltd.
12. Calculus, Mohamd Arif, S.Narayanan, T.K.Manicavachagon Pillay, S.Viswanathan Printers and Publishers Pvt. Ltd.
13. Differential and Integral Calculus, N.Piskunov, Mir Publishers, Moscow
14. Differential and Integral Calculus,Schamum's Outline Series,Frank Ayres
15. Calculus (Volume I & II), Tom.M.Apostol, John Wiley Publications
16. Calculus : An Historical Approach, W.M, Priestly (Springer)
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18. Application of Differentiation, S.Narayanan, T.K.Manicavachagon Pillay, , S.Viswanathan Printers and Publishers Pvt. Ltd.
19. Application of Differentiation,P.N.Arora, S.Arora, S.Chand & Company
20. Financial Mathematics, O.P.Malhotra, S.K.Gupta, Anubhuti Gangal, S.Chand & Company
21. Financial Mathematics ,Kashyap Trivedi, Chirag Trivedi, Pearson India Education Services Pvt. Ltd
22. Descriptive Statistics, Richard I.Levin, David S.Rubin, Prentice Hall Inc,Englewood, N.J.U.S.A
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24. Descriptive Statistics, Anderson, Sweenas, Williams, Library of Congress Cataloging in Publication Data
25. Correlation and Regression Analysis, Dr.S.P.Gupta, P.K. Gupta, Dr.Manmohan, Sultan Chand & Sons
26. Correlation and Regression Analysis, John.S.Croucher, Mc Graw-Hill Australia Pvt Limited
27. Operations Research, Dr.S.P.Gupta, P.K. Gupta, Dr.Manmohan, Sultan Chand & Sons
28. Operations Research, A.Ravindran, James J.Solberg, Willey Student Edition
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